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MASTER THESIS

Proof Theory of Circular Description Logics

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"Beauty is Nature in perfection; circularity is its chief attribute. Behold the full moon, the enchanting golf ball, the domes of splendid temples, the huckleberry pie, the wedding ring, the circus ring, the ring for the waiter, and the 'round' of drinks."

O. Henry

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Abstract

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In [12], Hofmann introduces a sequent calculus for the description logic $\mathcal{EL}$ where the TBoxes allow for circular concept definitions. In this thesis, we apply this framework to a family of DLs which allow for circular TBoxes. We start off by adjusting the calculus for the class of frame-based DLs, with base logic $\mathcal{FL}_0$. We continue to the attribute languages $\mathcal{AL}$ and $\mathcal{ALE}$. For these calculi, we prove soundness and completeness with respect to greatest fixpoint semantics for their fragments $\mathcal{AL}'$ and $\mathcal{ALE}'$, using Hofmann's strategy. When applying the theory to a logic including disjunction, we require a different strategy and introduce the notion of pre-interpretation for the soundness and completeness proofs. Then, we acknowledge the drawbacks of considering only greatest fixpoint semantics when it comes to circular definitions, and propose a calculus that includes greatest- and least-fixpoint semantics in one. Finally, we discuss the drawbacks of Hofmann's system for the classical logic $\mathcal{ALC}$ by discussing the consequences for the proofs when including the disjunction or negation, and give suggestions for future research.

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[1] Babies is the nickname given to the first year students, there were no actual babies harmed in the making of this thesis

Contents

Chapter 1

Introduction

In the development of intelligent systems using a large amount of data, there is a high need for logics that are not only decidable but have efficient reasoning algorithms. *Description Logic* (DL) is a family of knowledge representation languages, developed exactly with this goal. DLs have a wide range of real-world applications, all in category of data information storage or reasoning. One can think of databases, software information systems, but mostly it is used as an *ontology*: classification and explanation of concepts. As an ontology, DLs can be applied to any field. We can consider an ontology describing the roles within a university, or within the biomedical field, astronomy, software development, and many more.

Unlike other semantic networks or frames, DLs use logical symbols and expressions, giving us the tools to reason in a structured and well-understood way. This property makes it interesting to study the proof theory of DL. In this thesis, we apply the well-known Gentzen sequent calculus to a family of DLs.

Before getting into the details, we first paint a picture of the field. We start an introduction to basic notions and formalisms, and give an overview of general proof techniques. Then we introduce the sequent calculus we build on for the rest of the thesis.

1.1 Description Logic

For the introduction to the field of Description Logic, the book [4] has been used as the main source of information.

As mentioned, we refer to DLs as a family of knowledge representation languages. Before moving on to the formal notation, let us give an intuition behind the idea. Consider the setting of a *university*. A university consists of teachers, students, courses, faculties, study programs, etc. In DL, we call these the 'concepts' or 'concept names'. These concepts are related to each other using so-called roles: a teacher *teaches* a course, a student *is enrolled in* a study program, etc. All these concepts are defined and collected in what we call a *knowledge base*. There are different ways to describe concepts:

1. By giving the literal definition of a concept: 'a teacher is a person who teaches a course'

2. By describing the context of a concept: 'a teacher is a subset of people working in a university and a subset of not being a student'

3. By asserting that individual names stand for instances of concepts: 'Alice is a teacher'

4. By relating individual names by roles: 'Alice teaches Set Theory'

We make a division between the way to describe the first two points of this list and the last two points. Points 1 and 2 are considered to be part of the *terminology* and concepts of this form are collected together in the TBox, while 3 and 4 are the *assertional* part of the knowledge base and concepts of this form are put in the ABox. Let us now move on to the formal part.

The definition of concepts is given in the TBox $\mathcal{T}$ of the knowledge base. These definitions are of the form $C \sqsubseteq D$ and $C = D$. C is a concept name, and each concept name is defined by a concept description, in this case D, given by the following grammar:

$$
\begin{aligned}
B ::= & C \text{ (propositional concept)} \\
& \mid \top \text{ (universal concept)} \\
& \mid \bot \text{ (bottom concept)} \\
& \mid A \sqcap A \text{ (intersection, or conjunction)} \\
& \mid A \sqcup A \text{ (union, or disjunction)} \\
& \mid \neg A \text{ (negation)} \\
& \mid \exists r.A \text{ (existential restriction)} \\
& \mid \forall r.A \text{ (value restriction)}
\end{aligned}
$$

Thus, in the definition $Teacher = Person \sqcap \exists teaches.Course$, the values $Teacher$, $Person$ and $Course$ are concept names and $Person \sqcap \exists teaches.Course$ is a concept description. Except propositional concept names, all concept names are given a concept description. We will often refer to concept names simply as 'concepts'.

This is the syntax for the DL called $\mathcal{ALC}$: *attribute language with complements*. A semantics for an $\mathcal{ALC}$ concept description is an interpretation $\mathcal{I}$ mapping concept names and descriptions to subsets of a nonempty domain $D^{\mathcal{I}}$:

1. $\mathcal{I}(\top) = D^{\mathcal{I}}$

2. $\mathcal{I}(\bot) = \emptyset$

3. $\mathcal{I}(C \sqcap D) = \mathcal{I}(C) \cap \mathcal{I}(D)$

4. $\mathcal{I}(C \sqcup D) = \mathcal{I}(C) \cup \mathcal{I}(D)$

5. $\mathcal{I}(\neg C) = D^{\mathcal{I}} - \mathcal{I}(C)$

6. $\mathcal{I}(\exists r.C) = \{d \in D^{\mathcal{I}} \mid \text{there is an } e \in D^{\mathcal{I}} \text{ with } (d,e) \in \mathcal{I}(r) \text{ and } e \in \mathcal{I}(C)\}$

7. $\mathcal{I}(\forall r.C) = \{d \in D^{\mathcal{I}} \mid \text{for all } e \in D^{\mathcal{I}} \text{ if } (d,e) \in \mathcal{I}(r) \text{ then } e \in \mathcal{I}(C)\}$

We say that a concept C is *satisfiable* with respect to a TBox $\mathcal{T}$, if there is an interpretation $\mathcal{I}$ of $\mathcal{T}$ such that there is an element $a \in D^{\mathcal{I}}$ and $a \in \mathcal{I}(C)$. We write $\mathcal{T} \vDash E \sqsubseteq F$, if any interpretation $\mathcal{I}$ for a TBox $\mathcal{T}$ gives us $\mathcal{I}(E) \subseteq \mathcal{I}(F)$.

The elements of an ABox $\mathcal{A}$ are of the form $a : B$, where a is an instance of B. For example, $Alice : Teacher$ asserts that Alice is a teacher. Expressions in $\mathcal{A}$ can also be of the form $(Alice, ProofTheory) : teaches$ expressing that $Alice$ teaches $ProofTheory$. Finally, the knowledge base $\mathcal{K}$ is defined as the pair $\mathcal{K} = (\mathcal{T}, \mathcal{A})$.

Although being developed independently of each other, in this interpretation, it becomes clear that there is a relationship between DL and modal logic. As described in [4], there is a direct translation f from concept descriptions in the DL $\mathcal{ALC}$ to modal logic. Since we consider multiple types of relations in DL, such as *teaches*, *attends*, we want a modal logic where the diamonds and boxes are labeled. Therefore, f maps concept description in $\mathcal{ALC}$ to formulas in the modal logic K_m.

1. $f(A) = p_A$ for concept names A, and propositional letters p,
2. $f(C \sqcap D) = f(C) \wedge f(D)$,
3. $f(C \sqcup D) = f(C) \vee f(D)$,
4. $f(\neg C) = \neg f(C)$,
5. $f(\forall r.C) = \Box_r f(C)$,
6. $f(\exists r.C) = \Diamond_r f(C)$

However, one of the aspects that separate DL from modal logic is the notion of TBoxes and ABoxes, providing a convenient syntax.

An interpretation for formulas in modal logic is represented in Kripke models, and we can relate our TBox and ABox to such a model. Regarding the TBox $\mathcal{T}$, we say that for each formula $C \sqsubseteq D \in \mathcal{T}$, the formula $\neg f(C) \vee f(D)$ must hold in each world of our Kripke structure. We express this with the universal modality U. This allows [6] to prove the following theorem:

Theorem 1.1.1. Let $\mathcal{T}$ be an $\mathcal{ALC}$ TBox and E, F $\mathcal{ALC}$ concepts. Then:

- F is satisfiable iff $f(F)$ is satisfiable.
- F is satisfiable with respect to $\mathcal{T}$ iff $\bigwedge_{C \sqsubseteq D \in \mathcal{T}} [U](f(C) \to f(D)) \wedge f(F)$ is satisfiable
- $\mathcal{T} \vDash E \sqsubseteq F$ iff $\bigwedge_{C \sqsubseteq D \in \mathcal{T}} [U](f(C) \to f(D)) \to [U](f(E) \to f(F))$ is valid

Like TBoxes, we create a correspondence for ABoxes with the logic K_m, but need the extension with *nominals* denoted by the operator @. As explained in [9], this extension is added to modal logic to obtain *hybrid logic* with the goal to express that certain statements are only true in exactly one possible world. As an example, consider the statement: "it is sunny on 21 May 2022". In this case we want the statement "it is sunny", to be true exactly in the world where "21 May 2022" is true. Thus, in hybrid logic, we say the formula $@_a \phi$ holds if and only if in the one world where

a holds, ϕ holds. In this case, a stands for "21 May 2022" and ϕ for "it is sunny". Thus, an assertion of the form $a : C$ corresponds to the modal formula $@_a f(C)$, and $(a, b) : r$ to $@_a \Diamond_r b$.

In conclusion, there is a way of translating DLs to K_m, so why not just focus on the proof theory of modal logics instead? Although this might work for logic $\mathcal{ALC}$, the beauty of DL is that there are many variations with all very different applications, and not all of them translate directly to modal logic. We now discuss a few variations and high-light which ones we consider in this thesis.

1.1.1 Variations

We start by introducing the so-called *light-weight* DLs, obtained by removing certain logical connectives. One of the minimal description logics that still has applications in various fields, such as life sciences, is the description logic $\mathcal{EL}$ short for *existential language*. The syntax of this logic is the following:

$$\begin{aligned} B ::= & C \text{ (propositional concept)} \\ & | \top \text{ (universal concept)} \\ & | B \sqcap B \text{ (intersection, or conjunction of two concepts)} \\ & | \exists r.B \text{ (value restriction)} \end{aligned}$$

There are many more small logics to consider: we can replace $\exists r.$ by $\forall r.$, and obtain the logic $\mathcal{FL}_0$. We can add an atomic negation, or $\bot$. Since all these logics are very small, but still expressible enough to be useful in certain ontologies, they are interesting to study.

In this thesis, we focus on light-weight DLs that contain certain combinations of connectives.

Aside from the number of connectives, there are other types of extensions. In [7], the option is explored to create a "super logic", in particular the logic $\mu\,\mathcal{ALCIO}_{fa}$, featuring the following extensions: fixpoints, inverse roles, nominals and functionality assertions over atomic roles.

The extension $\mathcal{I}$ stands for inverse roles and allows us to invert the roles in our TBox: instead of adding $isChildOf$, we take the inverse of the $isParentOf$ relation. Then, the extension $\mathcal{O}$ stands for the addition of nominals and allows us to use ABox names, the a in $a : C$, within concept descriptions:

$$CoursesOfBob = Course \sqcap \exists taughtBy.\{Bob\}$$

Then fa stands for functional assertions, where the atomic roles r are functional. This means an ABox can not contain both $(a, b) : r$ and $(a, c) : r$, if b and c refer to two different elements of the domain, i.e. $\mathcal{I}(b) \neq \mathcal{I}(c)$.

The extension we consider in this thesis is the fixpoint extension, represented by the μ. Fixpoints can be incorporated by allowing for cyclic definitions in the TBox, e.g.:

$$Human = Mammal \sqcap \exists hasParent.\top \sqcap \forall hasParent.Human$$

While certain combinations of these extensions are still decidable, the "super logic" $\mu\mathcal{ALCIO}_{fa}$ is not. The battle of expressivity versus decidability is a common problem within the world of Description Logics.

1.1.2 Proof Theory

We now dive into the proof theory of Description Logic and sketch the work that has been done. The common reasoning problems that are considered in this proof theory are the following.

Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be an $\mathcal{ALC}$ knowledge base, and C, D $\mathcal{ALC}$ concepts, and b an individual name assertion. We say that

1. C is *satisfiable* with respect to $\mathcal{T}$ if there exists an interpretation $\mathcal{I}$ of $\mathcal{T}$ and some $d \in D^{\mathcal{I}}$ with $d \in \mathcal{I}(C)$;

2. C is *subsumed* by D with respect to $\mathcal{T}$, written $\mathcal{T} \vDash C \sqsubseteq D$, if $\mathcal{I}(C) \subseteq \mathcal{I}(D)$ for every interpretation $\mathcal{I}$ of $\mathcal{T}$;

3. C and D are *equivalent* with respect to $\mathcal{T}$, written $\mathcal{T} \vDash C = D$, if $\mathcal{I}(C) = \mathcal{I}(D)$ for every interpretation $\mathcal{I}$ of $\mathcal{T}$;

4. $\mathcal{K}$ is *consistent* if there exists an interpretation of $\mathcal{K}$;

5. b is an *instance of* C with respect to $\mathcal{K}$, written $\mathcal{K} \vDash b : C$, if $\mathcal{I}(b) \in \mathcal{I}(C)$ for every interpretation $\mathcal{I}$ of K.

A sequent calculus would primarily solve item 2 on this list, where the sequents proved are of the form $C \sqsubseteq D$, the soundness would give us $\mathcal{I}(C) \subseteq \mathcal{I}(D)$. The equivalence described in point 3 would be achieved by proving $C \sqsubseteq D$ and $D \sqsubseteq C$. Before we dive into these, let us focus on other algorithms often used in DL.

As mentioned in the previous section, there exist expressive DLs as well as light-weight DLs, both of which benefit from different proof calculi. For the more expressive DLs, we introduce a tableau-based algorithm that checks knowledge base consistency, and for the light-weight DLs we describe a consequence-based reasoning algorithm.

First, we describe the tableau algorithm. In the first step, the ABox $\mathcal{A}$ is saturated based on the connectives of the concept descriptions in it. For example, if $a : C \sqcap D \in A$, then the set $\{a : C, a : D\}$ is added to $\mathcal{A}$, etc. This procedure continues until all connectives have been considered, or until a clash is found, i.e., a subset of the form $\{a : B, a : \neg B\}$. While this algorithm essentially checks the consistency of the knowledge base, it can be used for other reasoning problems. Take, for example, the subsumption problem, point 2: $C \sqsubseteq D$. Then, a concept C is subsumed by a concept

D with respect to $\mathcal{K}$ iff $(\mathcal{T}, \mathcal{A} \cup \{x : C \sqcap \neg D\})$ is not consistent (where x is new). An overview of tableaux for different extensions of $\mathcal{ALC}$ can be found in [5].

The above tableau algorithm works because it looks for a clash. However, we have seen the description logic $\mathcal{EL}$, that contains neither $\neg$ nor $\bot$ making it impossible to derive a clash. Therefore, knowledge base consistency is an entirely trivial problem. Instead, one considers consequence-based reasoning for smaller, negation-free description logics.

The goal of the algorithm is simply to generate consequences, i.e., sequents of the form $A \sqsubseteq B$. The first step is to normalize the TBox, so that we only have sequents of the following form:

$$A \sqsubseteq B, A_1 \sqcap A_2 \sqsubseteq B, A \sqsubseteq \exists r.B, \exists r.A \sqsubseteq B$$

where A, A_1, A_2, B are propositional concept names or $\top$, and r is a role name. This is also the form of $\mathcal{T}$-sequent. Then, we have rules that only apply to sequents of that form, which are called classification rules:

$$\frac{}{A \sqsubseteq A}\ CR1 \qquad \frac{}{A \sqsubseteq \top}\ CR2 \qquad \frac{A_1 \sqsubseteq A_2 \quad A_2 \sqsubseteq A_3}{A_1 \sqsubseteq A_3}\ CR3$$

$$\frac{A \sqsubseteq A_1 \quad A \sqsubseteq A_2 \quad A_1 \sqcap A_2 \sqsubseteq B}{A \sqsubseteq B}\ CR4$$

$$\frac{A \sqsubseteq \exists r.A_1 \quad A_1 \sqsubseteq B_1 \qquad \exists r.B_1 \sqsubseteq B}{A \sqsubseteq B}\ CR5$$

The soundness of completeness of these rules are stated below and the proofs of these can be found in [6].

Lemma 1.1.2 (Soundness)**.** If all the elements of a TBox $\mathcal{T}'$ follow from $\mathcal{T}$ and the $\mathcal{T}$-sequents above the line of one of the inference rules belong to $\mathcal{T}'$, then the $\mathcal{T}$-sequent below the line also follows from $\mathcal{T}$.

Lemma 1.1.3 (Completeness)**.** Let $\mathcal{T}$ be a general $\mathcal{EL}$ TBox in normal form and $\mathcal{T}^*$ the saturated TBox obtained by exhaustive application of the inference rules. Then $\mathcal{T} \models A \sqsubseteq B$ implies $A \sqsubseteq B \in \mathcal{T}^*$.

Clearly, this algorithm is different from the tableau algorithm described before, which is a refutation procedure instead of a generation procedure. One drawback of the latter algorithm is that proof search is very difficult since all the non-axiomatic rules cut one or multiple formulas to get the desired sequent.

A Gentzen-like sequent calculus would benefit both types of logic. Multiple sequent calculi have been developed for DL. We focus on the work carried out by [12].

In this work, a sequent calculus for the logic $\mathcal{EL}$ is introduced in which the TBox is allowed to be cyclic.

1.2 Hofmann's Sequent Calculus

In this section, we describe the sequent calculus introduced in [12]. Concept descriptions are given by formulas ϕ, and the TBox is a list of equations of the form $X = \phi(X)$. Given a TBox $\mathcal{T}$, the syntax is the following:

$$\phi ::= X \mid P \mid \phi \sqcap \psi \mid \exists r.\phi$$

Above, P ranges over a finite set of propositional concept names used in $\mathcal{T}$ and r over a finite set of defined role names. Infinite sets are not considered and left out of this thesis.

For the equations $X = \phi(X)$ of a given TBox $\mathcal{T}$, we have that each variable occurs at most once on the left hand side of an equation. The formula $\phi(X)$ may depend on X, allowing for circular definitions, and it may involve variables appearing as some left hand side in $\mathcal{T}$.

The corresponding interpretation $\mathcal{I}$ is a function from formulas to subsets of a domain $D^{\mathcal{I}}$, such that:

1. $\mathcal{I}(X) = \mathcal{I}(\phi(X))$
2. $\mathcal{I}(\phi \sqcap \psi) = \mathcal{I}(\phi) \cap \mathcal{I}(\psi)$
3. $\mathcal{I}(\exists r.\phi) = \{x \mid \exists y \in \mathcal{I}(\phi).\, \mathcal{I}(r)(x,y)\}$

We refer to this interpretation as the *descriptive semantics*, and we write $\phi \vDash_{des} \psi$ if $\mathcal{I}(\phi) \subseteq \mathcal{I}(\psi)$ for every interpretation $\mathcal{I}$. Let us now move on to the calculus.

The sequents are of the form $\phi \sqsubseteq \psi$, where $\sqsubseteq$ is a syntactic operator. We want a system such that the judgement $\phi \sqsubseteq \psi$ is derivable if and only if $\phi \vDash_{des} \psi$. The rules are as follows:

$$\frac{}{\phi \sqsubseteq \phi}\ Ax \qquad \frac{\phi \sqsubseteq \rho}{\phi \sqcap \psi \sqsubseteq \rho}\ \sqcap L_1 \qquad \frac{\psi \sqsubseteq \rho}{\phi \sqcap \psi \sqsubseteq \rho}\ \sqcap L_2$$

$$\frac{\phi \sqsubseteq \psi \quad \phi \sqsubseteq \rho}{\phi \sqsubseteq \psi \sqcap \rho}\ \sqcap R \qquad \frac{\phi \sqsubseteq \psi}{\exists r.\phi \sqsubseteq \exists r.\psi}\ \exists$$

$$\frac{\phi(X) \sqsubseteq \psi}{X \sqsubseteq \psi}\ DefL \qquad \frac{\psi \sqsubseteq \phi(X)}{\psi \sqsubseteq X}\ DefR$$

Using this calculus, Hofmann goes on to prove soundness and completeness with respect to the descriptive semantics. The proof is relatively straightforward, using induction on the rules for soundness, and creating a canonical model for the completeness proof.

However, there is a slight problem regarding this interpretation, and that is caused by the circular TBoxes. We sketch this problem with an example: $X = P \sqcap \exists r.X$, then given an interpretation of propositional concept name P, the assignment of $\mathcal{I}(X)$ is not unique. A possible assignment is $\mathcal{I}(X) = \emptyset$, since we then have $\mathcal{I}(X) = \mathcal{I}(P) \cap \mathcal{I}(\exists r.X)$ for any assignment of $\mathcal{I}(P)$. Another possible interpretation for X is an isolated circle, e.g. for an assignment $\mathcal{I}(X) = \{a, b, c, d\} \subseteq \mathcal{I}(P) \subseteq D^{\mathcal{I}}$, we have $\mathcal{I}(r) = \{(a, b), (b, c), (c, d), (d, a)\}$. Furthermore, $\mathcal{I}(X)$ can issue an infinite path, where $\mathcal{I}(X) = \{a_i \mid i \in \mathbf{N}\} \subseteq \mathcal{I}(P) \subseteq D^{\mathcal{I}}$, and $\mathcal{I}(r)(x, y) = \{(a_0, a_1), (a_1, a_2), (a_2, a_3), \ldots\}$. In order to assign a unique assignment to a circular definition in the TBox, Hofmann considers greatest fixpoint semantics.

Definition 1.2.1. An interpretation of a TBox under *greatest fixpoint semantics* is an interpretation $\mathcal{I}$ which has the further property that whenever $\mathcal{J}$ is a function mapping formulas over the TBox to subsets of $D^{\mathcal{I}}$ in such a way that:

1. $\mathcal{J}(P) = \mathcal{I}(P)$
2. $\mathcal{J}(\phi \sqcap \psi) = \mathcal{J}(\phi) \cap \mathcal{J}(\psi)$
3. $\mathcal{J}(X) \subseteq \mathcal{J}(\phi(X))$
4. $\mathcal{J}(\exists r.\phi) = \{x \mid \exists y \in \mathcal{J}(\phi). \mathcal{I}(r)(x, y)\}$

then $\mathcal{J}(\phi) \subseteq \mathcal{I}(\phi)$ for all ϕ.

We write $\phi \vDash_{gfp} \psi$ to mean that $\mathcal{I}(\phi) \subseteq \mathcal{I}(\psi)$ for all greatest fixpoint interpretations $\mathcal{I}$.

The first two points are fairly straightforward. We only want to consider interpretations with the same assignment for propositional concept names, and the conjunction of two formulas is still the intersection of the interpretations of both formulas. For point 3, we consider only the functions such that $\mathcal{J}(X) \subseteq \mathcal{J}(\phi(X))$, since we need all functions $\mathcal{J}$ to be fixpoint functions. Point 4 uses the interpretation $\mathcal{I}$ to interpret the role, instead of $\mathcal{J}$. The reason is that we only compare interpretations where the interpretation of the roles are the same.

Given a fixed domain and an assignment to the propositional concept names and role names, the greatest fixpoint interpretation gives us a unique interpretation for a TBox $\mathcal{T}$ with circular definitions. Considering our previous example, $X = P \sqcap \exists r.X$, the greatest fixpoint interpretation is the option where $\mathcal{I}(X)$ issues an infinite path. The interpretation is now uniquely determined by the values assigned to the propositional concept names. Furthermore, for a TBox containing the equations: $X = P \sqcap \exists r.X$ and $Y = P \sqcap \exists r.Y$, $X \vDash_{gfp} Y$ is true while $X \vDash_{des} Y$ is not.

There is one thing we still need to fix in our system. The natural way to prove the judgement $X \sqsubseteq Y$ would lead to an infinite proof tree, as seen in the following example.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{X \sqsubseteq Y}}{\exists r.X \sqsubseteq \exists r.Y}\;\exists}{P \sqcap \exists r.X \sqsubseteq \exists r.Y}\;\sqcap L_2 \qquad \dfrac{\dfrac{}{P \sqsubseteq P}\;Ax}{P \sqcap \exists r.X \sqsubseteq P}\;\sqcap L_1}{\dfrac{P \sqcap \exists r.X \sqsubseteq P \sqcap \exists r.Y}{X \sqsubseteq P \sqcap \exists r.Y}\;DefL}\;\sqcap R}{X \sqsubseteq Y}\;DefR
$$

In order to avoid this, [12] defines a family of relations $\sqsubseteq_n$ for $n \in \mathbf{N}$:

1. $\phi \sqsubseteq_0 \psi$ for all ϕ, ψ
2. $\phi \sqsubseteq_n \phi$ for all $n \in \mathbf{N}$
3. The relations $\sqsubseteq_n$ are closed under the rules $\sqcap L_1, \sqcap L_2, \sqcap R, \forall$ and $DefL$
4. If $\phi \sqsubseteq_n \psi(X)$ then $\phi \sqsubseteq_{n+1} X$.

We write $\phi \sqsubseteq_\infty \psi$ to mean that $\phi \sqsubseteq_n \psi$ holds for all $n \in \mathbf{N}$. We note that whenever $\phi \sqsubseteq_n \psi$ and $n > m$ then $\phi \sqsubseteq_m \psi$.

The intuition behind the family of relations $\sqsubseteq_n$ is based on the searching procedure for the greatest fixpoint, i.e. a solution to the equation $\mathcal{I}(X) = \mathcal{I}(\phi(X))$. In this procedure, we start by evaluating whether $D^\mathcal{I}$ is a fixpoint and check whether $D^\mathcal{I} = \mathcal{I}(\phi(X \mapsto D^\mathcal{I})$. This starting point corresponds to the rule *start*, since $\phi \sqsubseteq_0 \psi$ holds for all ϕ, ψ, and $\mathcal{I}(\phi) \subseteq D^\mathcal{I}$ holds for all ϕ. If $D^\mathcal{I}$ is not a fixpoint, if $D^\mathcal{I} \supset \mathcal{I}(\phi(X \mapsto D^\mathcal{I})$, we try the output of $\phi(X \mapsto D^\mathcal{I})$ and check whether this is a fixpoint. Then, $\phi \sqsubseteq_\infty X$ corresponds to the finding of a fixpoint such that $\mathcal{I}(\phi) \subseteq \mathcal{I}(X)$ and $\mathcal{I}(\phi) \subseteq \mathcal{I}(\psi(X))$, and $\mathcal{I}(X) = \mathcal{I}(\phi(X))$. If we can not find a fixpoint, then there is an n such that $\phi \sqsubseteq_n X$, but not $\phi \sqsubseteq_{n+1} X$. We continue explaining this intuition in depth in chapter 5.

Incorporating this family of relations in our proof system gives us the following.

$$
\dfrac{}{\phi \sqsubseteq_0 \psi}\;start \qquad \dfrac{}{\phi \sqsubseteq_n \phi}\;Ax \qquad \dfrac{\phi \sqsubseteq_n \rho}{\phi \sqcap \psi \sqsubseteq_n \rho}\;\sqcap L_1
$$

$$
\dfrac{\psi \sqsubseteq_n \rho}{\phi \sqcap \psi \sqsubseteq_n \rho}\;\sqcap L_2 \qquad \dfrac{\phi \sqsubseteq_n \psi \quad \phi \sqsubseteq_n \rho}{\phi \sqsubseteq_n \psi \sqcap \rho}\;\sqcap R \qquad \dfrac{\phi \sqsubseteq_n \psi}{\exists r.\phi \sqsubseteq_n \exists r.\psi}\;\exists
$$

$$
\dfrac{\phi(X) \sqsubseteq_n \psi}{X \sqsubseteq_n \psi}\;DefL \qquad \dfrac{\psi \sqsubseteq_n \phi(X)}{\psi \sqsubseteq_{n+1} X}\;DefR
$$

In this thesis, we will often write $\phi \sqsubseteq_n \psi$ to mean "there is a proof for the judgement $\phi \sqsubseteq_n \psi$".

To show that it is not trivial that $\phi \sqsubseteq_n \psi$ has to hold for every $n \in \mathbf{N}$, consider the following example. Take $X = P \sqcap \exists r.X$ and $Y = P \sqcap \exists r.(P \sqcap \exists r.P)$. Then, there is a proof $Y \sqsubseteq_3 X$, but not for $Y \sqsubseteq_4 X$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P \sqsubseteq_2 P}\,Ax}{P \sqcap \exists r.(P \sqcap \exists r.P) \sqsubseteq_2 P}\,\sqcap L_1 \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \sqsubseteq_1 P}\,Ax}{P \sqcap \exists r.P \sqsubseteq_1 P}\,\sqcap L_1 \quad \dfrac{\dfrac{\dfrac{\dfrac{}{P \sqsubseteq_0 P \sqcap \exists r.X}\,start}{P \sqsubseteq_1 X}\,DefR}{\exists r.P \sqsubseteq_1 \exists r.X}\,\exists}{P \sqcap \exists r.P \sqsubseteq_1 \exists r.X}\,\sqcap L_2}{P \sqcap \exists r.P \sqsubseteq_1 P \sqcap \exists r.X}\,\sqcap R}{P \sqcap \exists r.P \sqsubseteq_2 X}\,DefR}{\exists r.(P \sqcap \exists r.P) \sqsubseteq_2 \exists r.X}\,\exists}{P \sqcap \exists r.(P \sqcap \exists r.P) \sqsubseteq_2 \exists r.X}\,\sqcap L_2}{\dfrac{P \sqcap \exists r.(P \sqcap \exists r.P) \sqsubseteq_2 P \sqcap \exists r.X}{\dfrac{Y \sqsubseteq_2 P \sqcap \exists r.X}{Y \sqsubseteq_3 X}\,DefR}\,DefL}\,\sqcap R$$

We sketch one more example to show that the infinite regress mentioned before is prevented. Consider $X = P \sqcap \exists r.X$ and $Y = P \sqcap \exists r.Y$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{X \sqsubseteq_0 P \sqcap \exists r.Y}\,start}{X \sqsubseteq_1 Y}\,DefR}{\exists r.X \sqsubseteq_1 \exists r.Y}\,\exists}{P \sqcap \exists r.X \sqsubseteq_1 \exists r.Y}\,\sqcap L_2 \quad \dfrac{\dfrac{}{P \sqsubseteq_1 P}\,Ax}{P \sqcap \exists r.X \sqsubseteq_1 P}\,\sqcap L_1}{\dfrac{P \sqcap \exists r.X \sqsubseteq_1 P \sqcap \exists r.Y}{\dfrac{X \sqsubseteq_1 P \sqcap \exists r.Y}{X \sqsubseteq_2 Y}\,DefR}\,DefL}\,\sqcap R$$

We proved $X \sqsubseteq_n Y$ for $n = 2$, but we could have continued this proof for any n.

In the rest of the paper, [12] continues to prove the soundness and completeness of the sequent calculus for $\mathcal{EL}$ with respect to the greatest fixpoint semantics. He then exchanges the existential quantifier with the universal quantifier and proves soundness and completeness of the introduced system with respect to descriptive semantics. He ends his work with a sketch of the full system for $\mathcal{ALC}$. It is stated that $\phi \sqsubseteq_\infty \psi$ iff $\phi \vDash_{gfp} \psi$, but this result is not proven.

1.3 Structure and goal

The goal of this thesis is to take the framework and strategy of [12] and apply it to light-weight DLs including circular definitions. We start with the family of frame-based description languages in chapter 2. In this chapter, we exchange the existential quantifier in the logic of [12] for the universal quantifier. The ideas in the proofs for this operator are inspired by Hofmann's soundness and completeness proofs for this logic with respect to the descriptive semantics but extended to work for the greatest fixpoint interpretation. This is also the first time that we present the strategy and all components needed for the proofs in detail.

Then, in chapter 3, we continue to do the same for the family of attribute languages. The operators atomic negation, $\exists r.\top$ and $\bot$ are added to obtain the language $\mathcal{AL}$. Only the operator $\bot$ requires a new rule, which we base on existing rules of sequent calculi, as presented, for example, in [13]. Then, we add the full existential quantification for $\mathcal{ALE}$. Since the two quantifiers are not interdefinable in these logics, we define the semantics in two separate relations.

In chapter 4, the disjunction $\sqcup$ is considered for a small logic only containing conjunction, $\bot$ and $\top$. We see that soundness and completeness can not be proved in the same way, and we need a new strategy. For this, we use some basic fixpoint ideas as presented in [11].

In chapter 5, we extend Hofmann's framework to a new framework including both greatest and least fixpoint operators, and we argue the benefits of such a calculus. In this chapter, we present the framework, but do not prove anything rigorous. We end with a discussion on the drawbacks of Hofmann's strategy for the language $\mathcal{ALC}$.

Chapter 2

Frame-based Description Languages

2.1 Introduction

In this chapter we apply the framework as introduced in [12], to the description logic $\mathcal{FL}_0$. The logic is built with the following constructs:

$$\phi ::= \top \mid P \mid X \mid \phi \sqcap \phi \mid \forall r.\phi$$

As in any logic we describe in this thesis, we allow circular definitions in our TBox, and the formulas of the TBox are of the form $X = \phi(X)$. We do not consider the case where two or more definitions depend on each other, i.e., $X = \phi(X, Y)$ and $Y = \psi(X, Y)$.

For the interpretation, we give the following definition:

Definition 2.1.1 (Interpretation)**.** An interpretation $\mathcal{I}$ for the logic $\mathcal{FL}_0$ is a function mapping formulas to subsets of a non-empty domain $D^{\mathcal{I}}$, according to the following rules:

1. $\mathcal{I}(\top) = D^{\mathcal{I}}$
2. $\mathcal{I}(X) = \mathcal{I}(\phi(X))$
3. $\mathcal{I}(\phi \sqcap \psi) = \mathcal{I}(\phi) \cap \mathcal{I}(\psi)$
4. $\mathcal{I}(\forall r.\phi) = \{x \in D^{\mathcal{I}} \mid \forall y. \mathcal{I}(r)(x, y) \rightarrow y \in \mathcal{I}(\phi)\}$

The logic $\mathcal{FL}_0$ is the base of the so-called 'frame-based languages' in description logic, forming the ground for $\mathcal{FL}$ and $\mathcal{FL}^-$. This is a family of logics not allowing any form of negation or bottom. $\mathcal{FL}^-$ is $\mathcal{FL}_0$ extended with $\exists r.\top$, making it possible to ensure for a certain concept to have a successor. Then $\mathcal{FL}$ is again an extension of $\mathcal{FL}^-$, where we allow role restrictions. A description of these can be found in [8].

The goal of this chapter is to introduce a sequent calculus for the logic $\mathcal{FL}_0$ and prove that it is sound and complete with respect to the greatest fixpoint semantics.

2.2 Sequent Calculus

In contrast to the chapter where we introduced Hofmann's framework, we now include the family of relations $\sqsubseteq_n$ immediately in the sequent calculus.

$$\frac{}{\Gamma, \phi \sqsubseteq_n \phi}\ Ax \qquad \frac{}{\Gamma \sqsubseteq_n \top}\ Ax\top \qquad \frac{}{\Gamma \sqsubseteq_0 \phi}\ start \qquad \frac{\Gamma, \phi_1, \phi_2 \sqsubseteq_n \psi}{\Gamma, \phi_1 \sqcap \phi_2 \sqsubseteq_n \psi}\ \sqcap L$$

$$\frac{\Gamma \sqsubseteq_n \psi_1 \quad \Gamma \sqsubseteq_n \psi_2}{\Gamma \sqsubseteq_n \psi_1 \sqcap \psi_2}\ \sqcap R \qquad \frac{\Gamma, \phi(X) \sqsubseteq_n \psi}{\Gamma, X \sqsubseteq_n \psi}\ DefL \qquad \frac{\Gamma \sqsubseteq_n \psi(X)}{\Gamma \sqsubseteq_{n+1} X}\ DefR \qquad \frac{\Gamma \sqsubseteq_n \phi}{\Lambda, \forall r.\Gamma \sqsubseteq_n \forall r.\phi}\ \forall$$

Let us consider a few notable aspects of this system. Firstly, we have (finite) sets of formulas, noted as Γ, on the left of the $\sqsubseteq_n$ symbol, and singular formulas on the right. This is needed for the rule $\sqcap L$. In [12], the rules for $\sqcap L$ are split into the following two rules:

$$\frac{\phi_1 \sqsubseteq_n \psi}{\phi_1 \sqcap \phi_2 \sqsubseteq_n \psi}\ \sqcap L_1 \qquad \frac{\phi_2 \sqsubseteq_n \psi}{\phi_1 \sqcap \phi_2 \sqsubseteq_n \psi}\ \sqcap L_2$$

The purpose is to allow distributivity of the universal quantifier over the conjunction. In our presented proof system, we can prove both $\forall r.(\phi \sqcap \psi) \sqsubseteq_n \forall r.\phi \sqcap \forall r.\psi$ for every n, as well as $\forall r.\phi \sqcap \forall r.\psi \sqsubseteq_n \forall r.(\phi \sqcap \psi)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\phi, \psi \sqsubseteq_n \phi}\ Ax}{\phi \sqcap \psi \sqsubseteq_n \phi}\ \sqcap L}{\forall r.(\phi \sqcap \psi) \sqsubseteq_n \forall r.\phi}\ \forall \quad \dfrac{\dfrac{\dfrac{}{\phi, \psi \sqsubseteq_n \psi}\ Ax}{\phi \sqcap \psi \sqsubseteq_n \psi}\ \sqcap L}{\forall r.(\phi \sqcap \psi) \sqsubseteq_n \forall r.\psi}\ \forall}{\forall r.(\phi \sqcap \psi) \sqsubseteq_n \forall r.\phi \sqcap \forall r.\psi}\ \sqcap R \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\phi, \psi \sqsubseteq_n \phi}\ Ax \quad \dfrac{}{\phi, \psi \sqsubseteq_n \psi}\ Ax}{\phi, \psi \sqsubseteq_n \phi \sqcap \psi}\ \sqcap R}{\forall r.\phi, \forall r.\psi \sqsubseteq_n \forall r.(\phi \sqcap \psi)}\ \forall}{\forall r.\phi \sqcap \forall r.\psi \sqsubseteq_n \forall r.(\phi \sqcap \psi)}\ \sqcap L$$

For the interpretation of a set of formulas Γ, appearing on the left of $\sqsubseteq_n$, we write $\mathcal{I}^{\sqcap}(\Gamma) = \bigcap\{\mathcal{I}(\gamma) \mid \gamma \in \Gamma\}$.

Furthermore, this calculus gives us the ability to weaken a judgement $\Gamma \sqsubseteq_n \psi$ to $\Gamma, \phi \sqsubseteq_n \psi$. Let us prove this:

Lemma 2.2.1 (Weakening)**.** If $\Gamma \sqsubseteq_n \psi$ is provable, then $\Gamma, \phi \sqsubseteq_n \psi$ is.

Proof. We prove this by induction on the length of the derivation $\Gamma \sqsubseteq_n \psi$. The base case is a derivation where only one rule has been used before weakening: *start*, *Ax* or *Ax*$\top$.

- *start*:

$$\frac{}{\Gamma \sqsubseteq_0 \psi}\ start$$

  Then the weakened judgement $\Gamma, \phi \sqsubseteq_0 \psi$ is just another instance of *start*.

- Ax:

$$\frac{}{\Gamma', \psi \sqsubseteq_n \psi}\ Ax$$

Then, the result we want to derive is another instance of Ax: $\Gamma', \phi, \psi \sqsubseteq_n \psi$.

$$\frac{}{\Gamma', \phi, \psi \sqsubseteq_n \psi}\ Ax$$

- $Ax\top$:

$$\frac{}{\Gamma \sqsubseteq_n \top}\ Ax\top$$

Again, the sequent we want to derive is an instance of $Ax\top$:

$$\frac{}{\Gamma, \phi \sqsubseteq_n \top}\ Ax\top$$

For every inductive step, we can just assume we can weaken the premises and therefore apply the rule to obtain the weakened conclusion. The only interesting case is when the last rule used is the $\forall$-rule:

- $\forall$:

$$\frac{\Gamma' \sqsubseteq_n \psi}{\Lambda, \forall r.\Gamma' \sqsubseteq_n \forall r.\psi}\ \forall$$

Since we can do this weakening within the $\forall$ rule, we can just add the formula ϕ there:

$$\frac{\Gamma' \sqsubseteq_n \psi}{\Lambda, \phi, \forall r.\Gamma' \sqsubseteq_n \forall r.\psi}\ \forall$$

□

We refer to this lemma in derivations in the following way:

$$\frac{\Gamma \sqsubseteq_n \psi}{\Gamma, \phi \sqsubseteq_n \psi}\ \textit{weakening}$$

Now let us move on to the greatest fixpoint semantics.

Definition 2.2.2 (Greatest fixpoint semantics)**.** An interpretation under the *greatest fixpoint semantics* for $\mathcal{FL}_0$ is an interpretation $\mathcal{I}$ which has the further property that whenever $\mathcal{J}$ is a function mapping the formulas over the TBox to subsets of the domain $D^{\mathcal{I}}$ in such a way that:

1. $\mathcal{J}(P) = \mathcal{I}(P)$
2. $\mathcal{J}(\top) = D^{\mathcal{I}}$

3. $\mathcal{J}(\phi \sqcap \psi) = \mathcal{J}(\phi) \cap \mathcal{J}(\psi)$

4. $\mathcal{J}(X) \subseteq \mathcal{J}(\phi(X))$

5. $\mathcal{J}(\forall r.\phi) = \{x \mid \forall y. \mathcal{I}(r)(x,y) \to y \in \mathcal{J}(\phi)\}$

then $\mathcal{J}(\phi) \subseteq \mathcal{I}(\phi)$ for all ϕ.

Further along, in the soundness and completeness proofs, we refer to these points 1-5 as the *conditions* for the interpretation that need to be fulfilled. Let us now start the soundness proof.

2.3 Soundness

First we introduce the following definition:

Definition 2.3.1. We write $\Gamma \Vdash \psi$ to denote that the judgement $\Gamma \sqsubseteq_n \psi$ can be derived with the rules Ax, $Ax\top$, $\sqcap L$ and $DefL$ for every $n \in \mathbf{N}$.

Since we have a derivation only using the axioms and the left rules, we can easily prove this adjusted version of the cut rule, that we call *Partial Cut*.

Lemma 2.3.2 (Partial Cut)**.** If $\Gamma \Vdash \psi$ and $\Delta, \psi \sqsubseteq_n \rho$, then $\Gamma, \Delta \sqsubseteq_n \rho$ for all $n \in \mathbf{N}$.

Proof. We do induction on the length of the derivation $\Gamma \Vdash \psi$. For the base case, we assume that the last rule used was Ax or $Ax\top$.

1. Take Ax as the last rule used:

   We assume both the judgements $\Gamma', \psi \Vdash \psi$ and $\Delta, \psi \sqsubseteq_n \rho$. Then we can just weaken the latter to get our wanted result: $\Gamma', \Delta, \psi \sqsubseteq_n \rho$.

2. Take $Ax\top$ as the last rule used:

   We assume $\Gamma \Vdash \top$ and $\Delta, \top \sqsubseteq_n \rho$. In this case, we need to consider two cases: either $\Delta, \top \sqsubseteq_n \rho$ is derived such that $\top$ is principal, or it is not.

   (a) If $\top$ is principal, then the last rule used is Ax, and $\rho = \top$. In this case, we can obtain our desired sequent by weakening $\Gamma \Vdash \top$ to $\Gamma, \Delta \sqsubseteq_n \top$.

   (b) If $\top$ is not principal, then let us say the last rule used was R such that:

   $$\frac{\Delta', \top \sqsubseteq_n \rho'}{\Delta, \top \sqsubseteq_n \rho}\ R$$

   We can then apply the lemma to $\Delta', \top \sqsubseteq_n \rho'$ to obtain $\Gamma, \Delta' \sqsubseteq_n \rho'$, and apply R to obtain: $\Gamma, \Delta \sqsubseteq_n \rho$.

For the inductive step, we assume that the last rule used is $\sqcap L$ or $DefL$, and we assume that we can cut the formula ψ before applying the rules $\sqcap L$ or $DefL$.

3. Take $\sqcap L$ as the last rule used:

   From the induction hypothesis and the assumption $\Delta, \psi \sqsubseteq_n \rho$, we have the judgement $\Gamma', \Delta, \phi_1, \phi_2 \sqsubseteq_n \rho$. Then by applying $\sqcap L$, we get the judgement: $\Gamma', \Delta, \phi_1 \sqcap \phi_2 \sqsubseteq_n \rho$.

4. Take $DefL$ as the last rule used:

   From the induction hypothesis and the derivation $\Delta, \phi(X) \sqsubseteq_n \rho$, we get the judgement $\Gamma', \Delta, \phi(X) \sqsubseteq_n \rho$. Then by applying $DefL$, we get the judgement: $\Gamma', \Delta, X \sqsubseteq_n \rho$

□

We now use this lemma in derivations as the following rule:

$$\frac{\Gamma \Vdash \psi \quad \Delta, \psi \sqsubseteq_n \rho}{\Gamma, \Delta \sqsubseteq_n \rho}\ pc$$

where *pc* stands for *Partial Cut*.

Before moving on to the next step, we explain the choice of the rules in the $\Vdash$ operator. The problem lies mostly in the $DefR$ rules. Let us assume that we do not have a definition of $\Vdash$, and just try to obtain admissibility of cut in our $\sqsubseteq_n$ operator. For one of the cases, we assume the following derivation:

$$\frac{\dfrac{\Gamma \sqsubseteq_n \phi(X)}{\Gamma \sqsubseteq_{n+1} X}\ DefR \quad \dfrac{\phi(X) \sqsubseteq_{n+1} \rho}{X \sqsubseteq_{n+1} \rho}\ DefL}{\Gamma \sqsubseteq_{n+1} \rho}\ pc$$

We want to transform this to a derivation, where the rule *pc* is pushed higher in the derivation. Using the fact that $\phi(X) \sqsubseteq_{n+1} \rho$ implies $\phi(X) \sqsubseteq_n \rho$, and the induction hypothesis, we get:

$$\frac{\Gamma \sqsubseteq_n \phi(X) \quad \phi(X) \sqsubseteq_n \rho}{\Gamma \sqsubseteq_n \rho}\ pc$$

Thus, we lose our step of moving on to the next natural number by applying $DefR$. We wanted to prove $\Gamma \sqsubseteq_{n+1} \rho$, but we proved $\Gamma \sqsubseteq_n \rho$. It might still be possible to admit a cut rule that includes $DefR$, but this requires a different solution, which we leave out of this thesis.

Having defined this rule, we move on to the *generation lemma*. The generation lemma is closely related to the well-known inversion lemma often mentioned in the field of proof theory; see [13]. We use the results of this lemma in both the soundness and the completeness proof.

Lemma 2.3.3 (Generation)**.** Suppose $n > 0$:

1. $\Gamma \sqsubseteq_n P$ iff $\Gamma \Vdash P$

2. $\Gamma \sqsubseteq_n \top$ iff $\Gamma \Vdash \top$
3. $\Gamma \sqsubseteq_n \psi_1 \sqcap \psi_2$ iff $\Gamma \sqsubseteq_n \psi_1$ and $\Gamma \sqsubseteq_n \psi_2$
4. $\Gamma \sqsubseteq_{n+1} X$ iff $\Gamma \sqsubseteq_n \phi(X)$
5. $\Gamma \sqsubseteq_n \forall r.\phi$ iff for the set Δ of subformulas of the TBox, $\forall r.\phi$, and Γ that contains all formulas δ such that $\Gamma \Vdash \forall r.\delta$, we have $\Delta \sqsubseteq_n \phi$.

Proof. We prove the left-to-right direction, noted by (a), by induction on the length of the derivation. We show per case what the possible last rule used is. The right-to-left direction, noted by (b), is proven by applying the appropriate sequent calculus rule.

1. (a) Assume $\Gamma \sqsubseteq_n P$.

   Since P is a propositional concept name and thus consists of no connectives, $\Gamma \sqsubseteq_n P$ is derived only using the rules Ax, $\sqcap L$ or $DefL$, and thus $\Gamma \Vdash P$.

   (b) Assume $\Gamma \Vdash P$.

   By definition of $\Vdash$, there is a derivation of the judgement $\Gamma \sqsubseteq_n P$ for all $n \in \mathbf{N}$.

2. (a) Assume $\Gamma \sqsubseteq_n \top$.

   Then the last rule used is Ax, $Ax\top$, $\sqcap L$, or $DefL$. All of these are in the definition of $\Vdash$, and therefore we get $\Gamma \Vdash \top$

   (b) Assume $\Gamma \Vdash \top$.

   By definition of $\Vdash$, there is a derivation of the judgement $\Gamma \sqsubseteq_n \top$ for all $n \in \mathbf{N}$.

3. (a) Assume $\Gamma \sqsubseteq_n \psi_1 \sqcap \psi_2$.

   The last rule used is then either Ax, $\sqcap L$, $DefL$ or $\sqcap R$.

   - Ax:

   $$\frac{}{\Gamma', \psi_1 \sqcap \psi_2 \sqsubseteq_n \psi_1 \sqcap \psi_2}\ Ax$$

     We can derive the following:

   $$\dfrac{\dfrac{}{\Gamma', \psi_1, \psi_2 \sqsubseteq_n \psi_1}\ Ax}{\Gamma', \psi_1 \sqcap \psi_2 \sqsubseteq_n \psi_1}\ \sqcap L \qquad \dfrac{\dfrac{}{\Gamma', \psi_1, \psi_2 \sqsubseteq_n \psi_2}\ Ax}{\Gamma', \psi_1 \sqcap \psi_2 \sqsubseteq_n \psi_2}\ \sqcap L$$

   - $\sqcap L$:

   $$\frac{\Gamma', \phi_1, \phi_2 \sqsubseteq_n \psi_1 \sqcap \psi_2}{\Gamma', \phi_1 \sqcap \phi_2 \sqsubseteq_n \psi_1 \sqcap \psi_2}\ \sqcap L$$

By induction hypothesis, we obtain the judgement $\Gamma', \phi_1, \phi_2 \sqsubseteq_n \psi_1$ and $\Gamma', \phi_1, \phi_2 \sqsubseteq_n \psi_2$. By applying the $\sqcap L$ to both judgements, we get the results we want.

- $DefL$:

$$\frac{\Gamma', \phi(X) \sqsubseteq_n \psi_1 \sqcap \psi_2}{\Gamma', X \sqsubseteq_n \psi_1 \sqcap \psi_2}\ DefL$$

From the IH, we conclude $\Gamma', \phi(X) \sqsubseteq_n \psi_1$ and $\Gamma', \phi(X) \sqsubseteq_n \psi_2$. By applying $DefL$ to both judgements, we derive the needed result.

- $\sqcap R$:

$$\frac{\Gamma \sqsubseteq_n \psi_1 \quad \Gamma \sqsubseteq_n \psi_2}{\Gamma \sqsubseteq_n \psi_1 \sqcap \psi_2}\ \sqcap R$$

Then, we obtain $\Gamma \sqsubseteq_n \psi_1$ and $\Gamma \sqsubseteq_n \psi_2$, from the antecedent of $\sqcap R$.

(b) Assume $\Gamma \sqsubseteq_n \psi_1$ and $\Gamma \sqsubseteq_n \psi_2$. Then, by simply applying the $\sqcap R$ rule, we obtain $\Gamma \sqsubseteq_n \psi_1 \sqcap \psi_2$.

4. (a) Assume $\Gamma \sqsubseteq_{n+1} X$.

The last rule used is Ax, $\sqcap L$, $DefL$ or $DefR$.

- Ax:
We assume $\Gamma', X \sqsubseteq_{n+1} X$. By our definition of the family of relations $\sqsubseteq_n$ we know $\Gamma', X \sqsubseteq_n X$. Then, we obtain the following derivation:

$$\frac{\dfrac{}{\Gamma', X \Vdash X}\ Ax \qquad \dfrac{\dfrac{}{\psi(X) \sqsubseteq_n \psi(X)}\ Ax}{X \sqsubseteq_n \psi(X)}\ DefL}{\Gamma', X \sqsubseteq_n \psi(X)}\ pc$$

- $\sqcap L$ and $DefL$: these rules are clear since we can prove this by using the induction hypothesis and applying the appropriate rule.
- $DefR$:

$$\frac{\Gamma \sqsubseteq_n \psi(X)}{\Gamma \sqsubseteq_{n+1} X}\ DefR$$

Then $\Gamma \sqsubseteq_n \psi(X)$, since it was derived in our step before.

(b) Assume $\Gamma \sqsubseteq_n \psi(X)$. Applying $DefR$ gives us $\Gamma \sqsubseteq_{n+1} X$.

5. (a) Assume $\Gamma \sqsubseteq_n \forall r.\psi$.

The last rule used is Ax, $\sqcap L$, $DefL$ or $\forall$.

- Ax:
We assume $\Gamma', \forall r.\psi \Vdash \forall r.\psi$. Then, it follows that $\psi \in \Delta$. Since $\psi \sqsubseteq_n \psi$, we obtain $\Delta \sqsubseteq_n \psi$, by weakening.

- $\sqcap L$:

$$\frac{\Gamma', \phi_1, \phi_2 \sqsubseteq_n \forall r.\psi}{\Gamma', \phi_1 \sqcap \phi_2 \sqsubseteq_n \forall r.\psi} \sqcap L$$

By induction $\Gamma', \phi_1, \phi_2 \Vdash \forall r.\delta$ for all $\delta \in \Delta$. By applying $\sqcap L$ it follows that $\Gamma', \phi_1 \sqcap \phi_2 \Vdash \forall r.\delta$, and $\Delta \sqsubseteq_n \psi$ still holds.

- $DefL$: we can use an argument identical to the case for $\sqcap L$.
- $\forall$:

$$\frac{\Lambda \sqsubseteq_n \psi}{\Gamma', \forall r.\Lambda \sqsubseteq_n \forall r.\psi} \forall$$

It follows that $\Gamma', \forall r.\Lambda \Vdash \forall r.\lambda$ for all $\lambda \in \Lambda$. Thus $\Lambda \subseteq \Delta$ and $\Delta \sqsubseteq_n \psi$.

(b) For the right-to-left direction, assume $\Delta = \{\delta_0, \dots \delta_k\}$. Then, we can obtain our wanted result with the following derivation:

$$\frac{\Gamma \Vdash \forall r.\delta_0 \quad \dots \quad \Gamma \Vdash \forall r.\delta_k \quad \dfrac{\Delta \sqsubseteq_n \psi}{\forall r.\Delta \sqsubseteq_n \forall r.\psi} \forall}{\Gamma \sqsubseteq_n \forall r.\psi} pc$$

Actually, we apply the pc rule separately to every instance of $\Gamma \Vdash \forall r.\delta_i$, but for simplicity, we write it down as above.

□

Before we move on to the soundness proof, there is one property we want to prove about the $\Vdash$ symbol, that will help us.

Lemma 2.3.4. If $\Gamma \Vdash \psi$ then $\mathcal{I}^{\sqcap}(\Gamma) \subseteq \mathcal{I}(\psi)$ for any interpretation $\mathcal{I}$.

Proof. This proof is done by induction on the length of the derivation $\Gamma \Vdash \phi$. The last rule used was either Ax, $Ax\top$, $\sqcap L$ or $DefL$.

- Assume the last rule used was Ax. Then $\Gamma = \Gamma' \cup \{\psi\}$, and it is evident that $\mathcal{I}(\psi) \cap \mathcal{I}^{\sqcap}(\Gamma') \subseteq \mathcal{I}(\psi)$.

- Assume the last rule used was $Ax\top$. Since $\mathcal{I}(\top) = D^{\mathcal{I}}$, it follows that we have $\mathcal{I}^{\sqcap}(\Gamma) \subseteq \mathcal{I}(\top)$ for all Γ.

- Assume the last rule used was $\sqcap L$, then

$$\frac{\Gamma', \phi_1 \Vdash \psi \quad \Gamma', \phi_2 \Vdash \psi}{\Gamma', \phi_1 \sqcap \phi_2 \Vdash \psi} \sqcap L$$

By induction hypothesis we have $\mathcal{I}^{\sqcap}(\Gamma') \cap \mathcal{I}(\phi_1) \subseteq \mathcal{I}(\psi)$ and $\mathcal{I}^{\sqcap}(\Gamma') \cap \mathcal{I}(\phi_2) \subseteq \mathcal{I}(\psi)$. It follows that $\mathcal{I}^{\sqcap}(\Gamma') \cap \mathcal{I}(\phi_1) \cap \mathcal{I}(\phi_2) \subseteq \mathcal{I}^{\sqcap}(\Gamma') \cap \mathcal{I}(\phi_1) \subseteq \mathcal{I}(\psi)$.

- Assume the last rule used was $DefL$, and thus

$$\frac{\Gamma', \phi(X) \sqsubseteq_n \psi}{\Gamma', X \sqsubseteq_n \psi} \; DefL$$

By IH, we know that $\mathcal{I}^{\sqcap}(\Gamma') \cap \mathcal{I}(\phi(X)) \subseteq \mathcal{I}(\psi)$. Since $\mathcal{I}(\phi(X)) = \mathcal{I}(X)$, we get $\mathcal{I}^{\sqcap}(\Gamma') \cap \mathcal{I}(X) \subseteq \mathcal{I}(\psi)$.

□

Now we have all the elements we need for the soundness proof.

Theorem 2.3.5 (Soundness)**.** $\Gamma \sqsubseteq_\infty \psi$ implies $\Gamma \vDash_{gfp} \psi$.

Proof. We want to show that for all Γ and ϕ and any interpretation $\mathcal{I}$ under the greatest fixpoint semantics we have:

$$\Gamma \sqsubseteq_\infty \phi \Rightarrow \mathcal{I}^{\sqcap}(\Gamma) \subseteq \mathcal{I}(\phi)$$

This is equivalent to proving that for each ϕ and interpretation $\mathcal{I}$ one has

$$\bigcup_{\Gamma \sqsubseteq_\infty \phi} \mathcal{I}^{\sqcap}(\Gamma) \subseteq \mathcal{I}(\phi)$$

Our strategy is to take $\mathcal{J}(\phi) := \bigcup_{\Gamma \sqsubseteq_\infty \phi} \mathcal{I}^{\sqcap}(\Gamma)$, and show that $\mathcal{J}$ satisfies the conditions in the greatest fixpoint definition. It is clear that $\mathcal{I}(\phi) \subseteq \mathcal{J}(\phi)$, since $\phi \sqsubseteq_\infty \phi$ for any ϕ.

1. First, we need to prove that $\mathcal{J}(P) = \mathcal{I}(P)$ for any interpretation $\mathcal{I}$.

   For the first direction, assume $x \in \mathcal{J}(P)$. Then according to the definition of $\mathcal{J}$, there is a Γ such that $\Gamma \sqsubseteq_\infty P$ and $x \in \mathcal{I}^{\sqcap}(\Gamma)$. Due to the generation lemma, we get $\Gamma \Vdash P$, and thus $x \in \mathcal{I}(P)$ by lemma 2.3.4.

   The other direction holds since $\mathcal{I}(P) \subseteq \mathcal{J}(P)$.

2. Then, we need to show that $\mathcal{J}(\top) = D^{\mathcal{I}}$.

   Take $x \in \mathcal{J}(\top)$. Then $x \in \mathcal{I}^{\sqcap}(\Gamma)$ for some Γ such that $\Gamma \sqsubseteq_n \top$. The generation lemma gives us $\Gamma \Vdash \top$, and therefore $x \in D^{\mathcal{I}}$ by lemma 2.3.4.

   The other direction holds since $\mathcal{I}(\top) \subseteq \mathcal{J}(\top)$.

3. Moving on to the condition $\mathcal{J}(\phi_1 \sqcap \phi_2) = \mathcal{J}(\phi_1) \cap \mathcal{J}(\phi_2)$.

   Take $x \in \mathcal{J}(\phi_1 \sqcap \phi_2)$, then there is a Γ such that $x \in \mathcal{I}^{\sqcap}(\Gamma)$ and $\Gamma \sqsubseteq_\infty \phi_1 \sqcap \phi_2$. Then the generation lemma gives us $\Gamma \sqsubseteq_\infty \phi_1$ and $\Gamma \sqsubseteq_\infty \phi_2$, and thus $x \in \mathcal{J}(\phi_1)$ and $\mathcal{J}(\phi_2)$, by definition of $\mathcal{J}$.

   Now consider $x \in \mathcal{J}(\phi_1) \cap \mathcal{J}(\phi_2)$, and thus $x \in \mathcal{J}(\phi_1)$ and $x \in \mathcal{J}(\phi_2)$. There are Γ_1, Γ_2 such that $\Gamma_1 \sqsubseteq_n \phi_1$ and $\Gamma_2 \sqsubseteq_n \phi_2$, and $x \in \mathcal{I}^{\sqcap}(\Gamma_1)$ and $x \in \mathcal{I}^{\sqcap}(\Gamma_2)$. We can obtain the following derivation:

$$\dfrac{\dfrac{\Gamma_1 \sqsubseteq_n \phi_1}{\Gamma_1, \Gamma_2 \sqsubseteq_n \phi_1}\ \textit{weakening} \quad \dfrac{\Gamma_2 \sqsubseteq_n \phi_2}{\Gamma_1, \Gamma_2 \sqsubseteq_n \phi_2}\ \textit{weakening}}{\Gamma_1, \Gamma_2 \sqsubseteq_n \phi_1 \sqcap \phi_2}\ \sqcap R$$

So since $x \in \mathcal{I}^{\sqcap}(\Gamma_1) \cap \mathcal{I}^{\sqcap}(\Gamma_2) = \mathcal{I}^{\sqcap}(\Gamma_1, \Gamma_2)$, we have $x \in \mathcal{J}(\phi_1 \sqcap \phi_2)$.

4. Next, we need to prove $\mathcal{J}(X) \subseteq \mathcal{J}(\phi(X))$.

   Take $x \in \mathcal{J}(X)$, then there is a Γ such that $x \in \mathcal{I}(\Gamma)$ and $\Gamma \sqsubseteq_\infty X$. By the generation lemma, we get $\Gamma \sqsubseteq_\infty \phi(X)$, and thus $x \in \mathcal{J}(\phi(X))$.

5. For the next case, we need to prove that $\mathcal{J}(\forall r.\phi) = \{x \mid \forall y \in D^{\mathcal{I}}. \mathcal{I}(r)(x,y) \rightarrow y \in \mathcal{J}(\phi)\}$.

   First assume $x \in \mathcal{J}(\forall r.\phi)$, then we get our Γ such that $\Gamma \sqsubseteq_\infty \forall r.\phi$, and $x \in \mathcal{I}^{\sqcap}(\Gamma)$. Since $\Gamma \sqsubseteq_\infty \forall r.\phi$, we know that for the set Δ such that $\Gamma \Vdash \forall r.\delta$ for all $\delta \in \Delta$, we have that $\Delta \sqsubseteq_\infty \phi$. Thus we know that $x \in \mathcal{I}(\forall r.\delta)$ for all $\delta \in \Delta$. Consider $y \in D^{\mathcal{I}}$ such that $\mathcal{I}(r)(x,y)$. Then we obtain $y \in \mathcal{I}(\delta)$ for all $\delta \in \Delta$, and thus $y \in \mathcal{I}^{\sqcap}(\Delta)$, and by definition of $\mathcal{J}$, $y \in \mathcal{J}(\phi)$.

   For the other direction, assume that if $\mathcal{I}(r)(x,y)$, then $y \in \mathcal{J}(\phi)$. Then for each y such that $\mathcal{I}(r)(x,y)$, we have $y \in \mathcal{I}^{\sqcap}(\Lambda)$ for some Λ such that $\Lambda \sqsubseteq_n \phi$. Then $x \in \mathcal{I}(\forall r.\lambda)$ for all $\lambda \in \Lambda$ and thus $x \in \mathcal{I}^{\sqcap}(\forall r.\Lambda)$. Applying the $\forall$ rule on $\Lambda \sqsubseteq_\infty \phi$ we get $\forall r.\Lambda \sqsubseteq_\infty \forall r.\phi$. Then $x \in \mathcal{J}(\forall r.\phi)$, as desired.

□

We have proven soundness and can now directly move on to completeness.

2.4 Completeness

Theorem 2.4.1 (Completeness)**.** If $\Gamma \vDash_{gfp} \psi$, then $\Gamma \sqsubseteq_\infty \psi$

Proof. Our strategy to prove this theorem is the following. We assume $\Gamma \vDash_{gfp} \psi$. Then we create an interpretation $\mathcal{I}$, such that $\mathcal{I}^{\sqcap}(\Gamma) \subseteq \mathcal{I}(\psi)$ implies $\Gamma \sqsubseteq_\infty \psi$ and prove that it is a greatest fixpoint interpretation. Because it is a greatest fixpoint interpretation, we know that $\Gamma \vDash_{gfp} \psi$ implies that $\mathcal{I}^{\sqcap}(\Gamma) \subseteq \mathcal{I}(\psi)$. This gives us the desired results.

Let us create the following interpretation $\mathcal{I}$:

$$\begin{aligned} D^{\mathcal{I}} &= \text{finite sets of subformulas of the TBox } \mathcal{T} \\ \mathcal{I}(\psi) &= \{\Gamma \mid \Gamma \sqsubseteq_\infty \psi\} \\ \mathcal{I}(r)(\Gamma, \Delta) &\leftrightarrow \Delta \text{ consists of all } \delta \text{ in the set} \\ &\qquad \text{of subformulas of } \mathcal{T} \text{ for which } \Gamma \Vdash \forall r.\delta \end{aligned}$$

Before continuing with the structure of the proof, it is important to note that the Δ in $\mathcal{I}(r)(\Gamma, \Delta)$ is always finite. The given TBox $\mathcal{T}$ is finite, and therefore the set of subformulas of this TBox $\mathcal{T}$ is also finite.

Now, the proof consists of two elements:

(a) Show that $\mathcal{I}$ satisfies the conditions for the fixpoint interpretation

(b) Show that $\mathcal{I}$ is indeed a greatest fixpoint interpretation: $\mathcal{J}(\phi) \subseteq \mathcal{I}(\phi)$ for all $\mathcal{J}$ and all ϕ. This follows from proving that if $\Gamma \in \mathcal{J}(\phi)$, then $\Gamma \sqsubseteq_\infty \phi$.

1. Let us consider the case of $\phi = P$

(a),(b) This case is trivial, since we consider only interpretations such that $\mathcal{I}(P) = \mathcal{J}(P)$.

2. Let us consider the case of $\phi = \top$

(a),(b) Also trivial, since $\Gamma \sqsubseteq_\infty \top$ is an axiom, and thus $\mathcal{J}(\top) = D^{\mathcal{I}} = \mathcal{I}(\top)$.

3. Let us consider the case of $\phi = \phi_1 \sqcap \phi_2$

   (a) We need to prove that $\mathcal{I}(\phi_1 \sqcap \phi_2) = \mathcal{I}(\phi_1) \cap \mathcal{I}(\phi_2)$.
   Take $\Gamma \in \mathcal{I}(\phi_1 \sqcap \phi_2)$, and therefore $\Gamma \sqsubseteq_\infty \phi_1 \sqcap \phi_2$. By the generation lemma, we obtain $\Gamma \sqsubseteq_\infty \phi_1$ and $\Gamma \sqsubseteq_\infty \phi_2$. Thus, $\Gamma \in \mathcal{I}(\phi_1)$ and $\Gamma \in \mathcal{I}(\phi_2)$ and thus $\Gamma \in \mathcal{I}(\phi_1) \cap \mathcal{I}(\phi_2)$.
   For the other direction assume $\Gamma \in \mathcal{I}(\phi_1)$ and $\Gamma \in \mathcal{I}(\phi_2)$. Then, $\Gamma \sqsubseteq_\infty \phi_1$ and $\Gamma \sqsubseteq_\infty \phi_2$, giving us $\Gamma \sqsubseteq_\infty \phi_1 \sqcap \phi_2$.

   (b) Then we show that if $\Gamma \in \mathcal{J}(\phi_1 \sqcap \phi_2)$ then $\Gamma \sqsubseteq_\infty \phi_1 \sqcap \phi_2$.
   Take $\Gamma \in \mathcal{J}(\phi_1 \sqcap \phi_2)$, then $\Gamma \in \mathcal{J}(\phi_1)$ and $\Gamma \in \mathcal{J}(\phi_2)$. Inductively, we may assume $\Gamma \sqsubseteq_\infty \phi_1$ and $\Gamma \sqsubseteq_\infty \phi_2$. We apply $\sqcap R$, and get $\Gamma \sqsubseteq_\infty \phi_1 \sqcap \phi_2$.

4. Let us consider the case of $\phi = X$

   (a) First, we need to prove that $\mathcal{I}(X) \subseteq \mathcal{I}(\psi(X))$.
   Take $\Gamma \in \mathcal{I}(X)$, then $\Gamma \sqsubseteq_\infty X$. The generation lemma then gives us $\Gamma \sqsubseteq_\infty \psi(X)$, and we conclude $\Gamma \in \mathcal{I}(\psi(X))$.

   (b) Then we show that if $\Gamma \in \mathcal{J}(X)$, then $\Gamma \sqsubseteq_n X$ for all $n \in \mathbf{N}$.
   Assume $\Gamma \in \mathcal{J}(X)$. We continue this proof by induction on n in $\Gamma \sqsubseteq_n X$. The base case $n = 0$ is trivial, since $\Gamma \sqsubseteq_0 X$ holds for all Γ and X.
   For the inductive step, we assume $\Gamma \sqsubseteq_n X$ and assume $\Gamma \in \mathcal{J}(X)$. Since $\mathcal{J}(X) \subseteq \mathcal{J}(\psi(X))$, it follows $\Gamma \in \mathcal{J}(\psi(X))$, and from the induction hypothesis, we obtain $\Gamma \sqsubseteq_n \psi(X)$. Then, by applying $Def R$, we obtain $\Gamma \sqsubseteq_{n+1} X$.

5. Let us consider the case of $\phi = \forall r.\phi_1$

(a) We need to prove that $\mathcal{I}(\forall r.\phi_1) = \{x \mid \forall y \in D^{\mathcal{I}}.\mathcal{I}(r)(x,y) \to y \in \mathcal{I}(\phi_1)\}$
Take $\Gamma \in \mathcal{I}(\forall r.\phi_1)$, then we know $\Gamma \sqsubseteq_\infty \forall r.\phi_1$. We assume a $\Delta \in D^{\mathcal{I}}$ such that $\mathcal{I}(r)(\Gamma, \Delta)$, then Δ consists of all δ such that $\Gamma \Vdash \forall r.\delta$, and by the generation lemma we get that $\Delta \sqsubseteq_\infty \phi_1$, and thus $\Delta \in \mathcal{I}(\phi_1)$.

Assume $\mathcal{I}(r)(\Gamma, \Delta)$ implies $\Delta \in \mathcal{I}(\phi_1)$. Take a Δ such that $\mathcal{I}(r)(\Gamma, \Delta)$. By the definition of $\mathcal{I}$ we get $\Delta \sqsubseteq_\infty \phi_1$, and by the definition of $\mathcal{I}(r)(\Gamma, \Delta)$ we get that Δ consists of all δ such that $\Gamma \Vdash \forall r.\delta$. From the generation lemma, we conclude $\Gamma \sqsubseteq_\infty \forall r.\phi_1$. This gives us $\Gamma \in \mathcal{I}(\forall r.\phi_1)$.

(b) Then we show that $\Gamma \in \mathcal{J}(\forall r.\phi_1)$ implies $\Gamma \sqsubseteq_\infty \forall r.\phi_1$.
Take $\Gamma \in \mathcal{J}(\forall r.\phi_1)$, and consider Δ such that $\mathcal{I}(r)(\Gamma, \Delta)$. By the definition of our model this gives us: $\Gamma \Vdash \forall r.\delta$ for all $\delta \in \Delta$, and we have $\Delta \in \mathcal{J}(\phi_1)$. Our induction hypothesis gives us that for any $\Lambda \in \mathcal{J}(\phi_1)$ have $\Lambda \sqsubseteq_\infty \phi_1$, and therefore $\Delta \sqsubseteq_\infty \phi_1$. The generation lemma gives us $\Gamma \sqsubseteq_\infty \forall r.\phi_1$.

Thus, we can conclude: if $\Gamma \models_{gfp} \psi$ then $\mathcal{I}^{\sqcap}(\Gamma) \subseteq \mathcal{I}(\phi)$. Since $\Gamma \in \mathcal{I}^{\sqcap}(\Gamma)$, we get $\Gamma \in \mathcal{I}(\phi)$, and this implies $\Gamma \sqsubseteq_\infty \phi$. $\square$

Chapter 3

Attribute Languages

3.1 Introduction

In this chapter, we dive into what is called the $\mathcal{AL}$-family of description logics, where $\mathcal{AL}$ stands for *attribute language*. The base language allows the following constructors:

$$\phi ::= \top \mid \bot \mid P \mid \neg P \mid X \mid \phi \sqcap \phi \mid \forall r.\phi \mid \exists r.\top$$

We still consider logics where the TBoxes have circular definitions, and therefore our syntax includes the symbol X. The interpretation of this logic is the following:

Definition 3.1.1 (Interpretation)**.** An interpretation $\mathcal{I}$ of the logic $\mathcal{AL}$ is a function mapping formulas to subsets of a non-empty domain $D^{\mathcal{I}}$, according to the following rules:

1. $\mathcal{I}(\top) = D^{\mathcal{I}}$
2. $\mathcal{I}(\bot) = \emptyset$
3. $\mathcal{I}(\neg P) = D^{\mathcal{I}} - \mathcal{I}(P)$
4. $\mathcal{I}(X) = \mathcal{I}(\phi(X))$
5. $\mathcal{I}(\phi \sqcap \psi) = \mathcal{I}(\phi) \cap \mathcal{I}(\psi)$
6. $\mathcal{I}(\exists r.\top) = \{x \mid \exists y \in D^{\mathcal{I}}.\mathcal{I}^1(r)(x,y)\}$
7. $\mathcal{I}(\forall r.\phi) = \{x \mid \forall y \in D^{\mathcal{I}}.\mathcal{I}^2(r)(x,y) \rightarrow y \in \mathcal{I}(\phi)\}$

There are a few interesting aspects to point out. First, we introduce a negation for the propositional concept names, referred to from now on as atomic negation, and a $\bot$. Thus far, in the frame-based languages, all subsumptions of the form $C \sqsubseteq D$ where satisfiable, i.e., there is always an interpretation $\mathcal{I}$ such that $\mathcal{I}(C) \subseteq \mathcal{I}(D)$. This is not the case anymore: there is no interpretation $\mathcal{I}$ that satisfies $\mathcal{I}(\top) \subseteq \mathcal{I}(\bot)$.

Furthermore, we now consider a limited existential quantifier and make concepts of the form:

$$Parent = Human \sqcap \exists hasChild.\top \sqcap \forall hasChild.Human$$

The purpose of formula $\exists hasChild.\top$ is to ensure that every parent has at least one child. Clearly, this is not as strong as having a full existential quantifier, but it is a significant addition to the language in terms of expressive power.

One thing to note about the interpretation $\mathcal{I}$ is the case for $\mathcal{I}(\exists r.\top)$ and $\mathcal{I}(\forall r.\phi)$. Since we only have an atomic negation and the existential restriction is limited to $\top$, the two quantifiers are not interdefinable. Thus, all interpretations for TBoxes containing concept descriptions with $\exists r$ and $\forall r.$ have two interpretations for the relation r. We denote the interpretation of the role of the existential restriction with $\mathcal{I}^1(r)(x,y)$, and for the universal quantifier with $\mathcal{I}^2(r)(x,y)$. However, this gives us just a fragment of $\mathcal{AL}$, namely the fragment where we have two sets of roles: the ones that occur existentially quantified and a set of roles that occur universally quantified, and no role appears in both sets. In this chapter, we only prove the results for this fragment of $\mathcal{AL}$, which we will refer to as $\mathcal{AL}'$.

From this logic, we can extend to the logic $\mathcal{AL}\,\mathcal{E}$, achieved by adding a full existential quantifier:

$$\phi ::= \top \mid \bot \mid P \mid \neg P \mid X \mid \phi \sqcap \phi \mid \forall r.\phi \mid \exists r.\phi$$

and the logic $\mathcal{ALC}$, also referred to as $\mathcal{AL}\,\mathcal{E}\,\mathcal{U}$:

$$\phi ::= \top \mid \bot \mid P \mid \neg P \mid X \mid \phi \sqcap \phi \mid \phi \sqcup \phi \mid \forall r.\phi \mid \exists r.\phi$$

In this chapter, we consider the above-mentioned fragments for the logics $\mathcal{AL}$ and $\mathcal{AL}\,\mathcal{E}$, referred to as $\mathcal{AL}'$ and $\mathcal{ALE}'$ respectively, and prove soundness and completeness with respect to the greatest fixpoint semantics.

3.2 $\mathcal{AL}'$

We start by considering the base language $\mathcal{AL}'$, given by:

$$\phi ::= \top \mid \bot \mid P \mid \neg P \mid X \mid \phi \sqcap \phi \mid \forall r.\phi \mid \exists r.\top$$

Let us introduce the sequent calculus, where we include the family of relations $\sqsubseteq_n$ in the proof system.

$$\frac{}{\Gamma, \phi \sqsubseteq_n \phi}\ Ax \qquad \frac{}{\Gamma \sqsubseteq_n \top}\ Ax\top \qquad \frac{}{\Gamma, \bot \sqsubseteq_n \phi}\ Ax\bot \qquad \frac{}{\Gamma \sqsubseteq_0 \phi}\ start$$

$$\frac{\Gamma, \phi_1, \phi_2 \sqsubseteq \psi}{\Gamma, \phi_1 \sqcap \phi_2 \sqsubseteq \psi}\ \sqcap L \qquad \frac{\Gamma \sqsubseteq_n \psi_1 \quad \Gamma \sqsubseteq_n \psi_2}{\Gamma \sqsubseteq_n \psi_1 \sqcap \psi_2}\ \sqcap R$$

$$\frac{\Gamma, \phi(X) \sqsubseteq_n \psi}{\Gamma, X \sqsubseteq_n \psi}\ DefL \qquad \frac{\Gamma \sqsubseteq_n \psi(X)}{\Gamma \sqsubseteq_{n+1} X}\ DefR \qquad \frac{\Gamma \sqsubseteq_n \phi}{\Lambda, \forall r.\Gamma \sqsubseteq_n \forall r.\phi}\ \forall$$

We note that the only rule added is the $Ax\bot$ rule. Rules for $\exists r.\top$ and $\neg P$ are not needed, which means formulas of these form can only be added by the axioms or by the weakening in the $\forall$ rule.

Otherwise, the proof system is identical to the one in the previous chapter, making the soundness and completeness proof straightforward to adjust. The interpretation of our syntax has been adjusted, and therefore we need to check whether these additions are sound and complete in our system.

Definition 3.2.1 (Greatest fixpoint semantics)**.** An interpretation under the greatest fixpoint semantics for $\mathcal{AL}'$ is an interpretation $\mathcal{I}$ which has the further property that whenever $\mathcal{J}$ is a function mapping the formulas over the TBox to subsets of the domain $D^{\mathcal{I}}$ in such a way that:

1. $\mathcal{J}(\top) = D^{\mathcal{I}} = \mathcal{I}(\top)$
2. $\mathcal{J}(\bot) = \emptyset = \mathcal{I}(\bot)$
3. $\mathcal{J}(P) = \mathcal{I}(P)$
4. $\mathcal{J}(\neg P) = D^{\mathcal{I}} - \mathcal{J}(P) = \mathcal{I}(\neg P)$
5. $\mathcal{J}(X) \subseteq \mathcal{J}(\phi(X))$
6. $\mathcal{J}(\phi \sqcap \psi) = \mathcal{J}(\phi) \sqcap \mathcal{J}(\psi)$
7. $\mathcal{J}(\exists r.\top) = \{x \mid \exists y \in D^{\mathcal{I}}.\mathcal{I}^1(r)(x,y)\}$
8. $\mathcal{J}(\forall r.\phi) = \{x \mid \forall y.\mathcal{I}^2(r)(x,y) \to y \in \mathcal{J}(\phi)\}$

then $\mathcal{J}(\phi) \subseteq \mathcal{I}(\phi)$ for all ϕ.

In the following sections, we do not go over all cases again, since those proofs are equivalent to the ones in the previous section.

3.2.1 Soundness

We make some minor adjustments necessary to the definition of $\Vdash$, by adding $Ax\bot$ to the rules used.

Definition 3.2.2. We write $\Gamma \Vdash \psi$ to denote that the judgement $\Gamma \sqsubseteq_n \psi$ can be derived with the rules Ax, $Ax\top$, $Ax\bot$, $\sqcap L$ and $DefL$ for every $n \in \mathbf{N}$.

We ensure that the partial cut lemma still holds.

Lemma 3.2.3 (Partial Cut)**.** If $\Gamma \Vdash \psi$ and $\Delta, \psi \sqsubseteq_n \rho$, then $\Gamma, \Delta \sqsubseteq_n \rho$ for all $n \in \mathbf{N}$.

Proof. The only change is the addition of the rule $Ax\bot$ to the definition of $\Vdash$. Therefore, we need to make sure this lemma still holds. We still perform an induction on the length of the derivation $\Gamma \Vdash \psi$, and we take the case where the last rule used is $Ax\bot$:

$$\frac{}{\Gamma', \bot \Vdash \psi} Ax\bot$$

We assume the sequent $\Delta, \psi \sqsubseteq_n \rho$, and we need to prove $\Gamma', \bot \sqsubseteq_n \rho$. However, this is an instance of the $Ax\bot$ axiom by itself, and so we are done. □

Then we move on to the *generation lemma*, where the new cases added are points 2, 4, and 7. The other rules need a minor adjustment, where we need to consider the last rule used to be an instance of $Ax\bot$

Lemma 3.2.4 (Generation)**.** Suppose $n > 0$:

1. $\Gamma \sqsubseteq_n \top$ iff $\Gamma \Vdash \top$
2. $\Gamma \sqsubseteq_n \bot$ iff $\Gamma \Vdash \bot$
3. $\Gamma \sqsubseteq_n P$ iff $\Gamma \Vdash P$
4. $\Gamma \sqsubseteq_n \neg P$ iff $\Gamma \Vdash \neg P$
5. $\Gamma \sqsubseteq_n \psi_1 \sqcap \psi_2$ iff $\Gamma \sqsubseteq_n \psi_1$ and $\Gamma \sqsubseteq_n \psi_2$
6. $\Gamma \sqsubseteq_{n+1} X$ iff $\Gamma \sqsubseteq_n \phi(X)$
7. $\Gamma \sqsubseteq_n \exists r.\top$ iff $\Gamma \Vdash \exists r.\top$
8. $\Gamma \sqsubseteq_n \forall r.\phi$ iff for the set Δ that contains all subformulas of the TBox and Γ and $\forall r.\delta$ such that $\Gamma \Vdash \forall r.\delta$ for all $\delta \in \Delta$, we have $\Delta \sqsubseteq_n \phi$.

Proof. 2. Assume $\Gamma \sqsubseteq_n \bot$.

The last rule used was Ax, $\sqcap L$, $DefL$ (in this case, $Ax\bot$ and Ax are the same):

- Ax:

$$\frac{}{\Gamma', \bot \sqsubseteq_n \bot} Ax$$

It follows directly that $\Gamma', \bot \Vdash \bot$.

- $\sqcap L$:

$$\frac{\Gamma', \phi, \psi \sqsubseteq_n \bot}{\Gamma', \phi \sqcap \psi \sqsubseteq_n \bot} \sqcap L$$

By IH we know $\Gamma', \phi, \psi \Vdash \bot$, and since $\sqcap L$ maintains the $\Vdash$ operator, we get $\Gamma', \phi \sqcap \psi \Vdash \bot$.

- $DefL$

$$\frac{\Gamma', \phi(X) \sqsubseteq \bot}{\Gamma', X \sqsubseteq \bot} DefL$$

By IH we know $\Gamma', \phi(X) \Vdash \bot$, and since $DefL$ maintains the $\Vdash$ operator, we get: $\Gamma', X \Vdash \bot$.

The proofs for 4 and 7 are obtained by replacing every instance of $\bot$ with $\neg P$ and $\exists r.\top$ respectively.

For the other items we need to consider the last rule used to be $\bot$. The only interesting case is point 8. For the other cases, the sequents that need to be obtained are just instances of $Ax\bot$ again.

8. Assume that the last rule used is $Ax\bot$ to obtain the sequent $\Gamma, \bot \sqsubseteq_n \forall r.\phi$. Then the set Δ containing all subformulas of the TBox, Γ, and $\exists r.\psi$ such that $\Gamma \Vdash \forall r.\delta$, is equal to the set of all subformulas of the TBox, Γ, $\exists r.\psi$. Thus, clearly $\Delta \sqsubseteq_n \phi$, since $\bot \in \Delta$.

□

The definition of $\Vdash$ has slightly changed, and therefore we need to add another case to the proof of the following lemma.

Lemma 3.2.5. If $\Gamma \Vdash \psi$ then $\mathcal{I}^{\sqcap}(\Gamma) \subseteq \mathcal{I}(\psi)$ for any interpretation $\mathcal{I}$.

Proof. We prove this by induction on the length of the derivation $\Gamma \Vdash \psi$. The last rule used is one of the following: Ax, $Ax\top$, $Ax\bot$, $\sqcap L$, or $DefL$. The only new one is $Ax\bot$, and we prove that the lemma still holds.

Assume that the last rule used is $Ax\bot$, and thus the sequent $\Gamma', \bot \Vdash \psi$ is derived. Since $\mathcal{I}(\bot) = \emptyset$, it follows that $\mathcal{I}^{\sqcap}(\Gamma') \cap \mathcal{I}(\bot) = \emptyset \subseteq \mathcal{I}(\psi)$ for all ψ. □

Now, we have all we need to complete the soundness proof.

Theorem 3.2.6 (Soundness)**.** $\Gamma \sqsubseteq_\infty \psi$ implies $\Gamma \vDash_{gfp} \psi$.

Proof. The strategy of this soundness proof is the same as before: we prove that for each ψ and interpretation $\mathcal{I}$ one has

$$\bigcup_{\Gamma \sqsubseteq_\infty \psi} \mathcal{I}^{\sqcap}(\Gamma) \subseteq \mathcal{I}(\psi)$$

Then, taking $\mathcal{J}(\phi) := \bigcup_{\Gamma \sqsubseteq_\infty \phi} \mathcal{I}^{\sqcap}(\Gamma)$, we show that $\mathcal{J}$ is a function that meets the conditions for the greatest fixpoint interpretation. We go over the cases where ψ is one of: $\bot$, $\exists r.\top$, and $\neg P$, since those are the new cases.

- Check $\mathcal{J}(\bot) = \emptyset$. Take $x \in \mathcal{J}(\bot)$, then there is a Γ such that $\Gamma \sqsubseteq_\infty \bot$, and $x \in \mathcal{I}^{\sqcap}(\Gamma)$. By the generation lemma we get $\Gamma \Vdash \bot$, and therefore $x \in \mathcal{I}(\bot) = \emptyset$. This gives us $\mathcal{J}(\bot) = \emptyset$.

- Check $\mathcal{J}(\neg P) = D^{\mathcal{I}} - \mathcal{J}(P)$.

  Take $x \in \mathcal{J}(\neg P)$, then there is a Γ such that $x \in \mathcal{I}^{\sqcap}(\Gamma)$ and $\Gamma \sqsubseteq_\infty \neg P$. By the generation lemma, $\Gamma \Vdash \neg P$, and thus $x \in \mathcal{I}(\neg P) = D^{\mathcal{I}} - \mathcal{I}(P) = D^{\mathcal{I}} - \mathcal{J}(P)$, by assumption on P.

For the other direction, assume $x \in D^{\mathcal{I}} - \mathcal{J}(P) = D^{\mathcal{I}} - \mathcal{I}(P) = \mathcal{I}(\neg P)$. Since the judgement $\neg P \sqsubseteq_\infty \neg P$ holds, we have that $\mathcal{I}(\neg P) \subseteq \mathcal{J}(\neg P)$, and thus $x \in \mathcal{J}(\neg P)$.

- Check $\mathcal{J}(\exists r.\top) = \{x \mid \exists y \in D^{\mathcal{I}}.\mathcal{I}^1(r)(x,y)\}$.

  Take $x \in \mathcal{J}(\exists r.\top)$, then there is Γ such that $x \in \mathcal{I}^{\sqcap}(\Gamma)$ and $\Gamma \sqsubseteq_\infty \exists r.\top$. By the generation lemma we get $\Gamma \Vdash \exists r.\top$, and thus $x \in \mathcal{I}(\exists r.\top)$. Thus, there is a y such that $\mathcal{I}^1(r)(x,y)$, and $y \in \mathcal{I}(\top)$. Since $\top \sqsubseteq_\infty \top$, we get $y \in \mathcal{J}(\top)$.

  Assume $x, y \in D^{\mathcal{I}}$ such that $\mathcal{I}^1(r)(x,y)$. Since $y \in \mathcal{I}(\top)$ we have $x \in \mathcal{I}(\exists r.\top)$. Since $\exists r.\top \sqsubseteq_\infty \exists r.\top$, we get $x \in \mathcal{J}(\exists r.\top)$.

□

3.2.2 Completeness

The completeness proof is where it becomes interesting. The strategy is the same: we create a greatest fixpoint interpretation and show that it implies $\Gamma \sqsubseteq_\infty \psi$. However, the model looks different. Since we want the interpretation of $\bot$ to be empty, we want to adjust our domain. Let us introduce the following notion:

Definition 3.2.7 (Consistency)**.** A set of formulas Γ is called consistent iff there is no proof $\Gamma \sqsubseteq_n \bot$ for any $n > 0$.

We adjust our domain to contain all the *consistent* sets of subformulas of the TBox $\mathcal{T}$.

Furthermore, we need to adjust our interpretation for r. We know have two operators addressing roles: $\exists r$ and $\forall r$. As already mentioned, the two restrictions are not interdefinable and thus need to be treated independently. We solve this by adding the following interpretations.

$$\begin{aligned}
\mathcal{I}^1(r)(\Gamma,\Delta) &\leftrightarrow \Gamma \Vdash \exists r.\top \\
\mathcal{I}^2(r)(\Gamma,\Delta) &\leftrightarrow \Delta \text{ consists of all } \delta \text{ in the set} \\
&\qquad \text{of subformulas of } \mathcal{T} \text{ for which } \Gamma \Vdash \forall r.\delta
\end{aligned}$$

We now move on to the completeness proof.

Theorem 3.2.8 (Completeness)**.** If $\Gamma \vDash_{gfp} \psi$ then $\Gamma \sqsubseteq_\infty \psi$

Proof. The strategy is the same as before: we create a greatest fixpoint interpretation $\mathcal{I}$, such that $\Gamma \Vdash_{gfp} \psi$ implies $\mathcal{I}^{\sqcap}(\Gamma) \subseteq \mathcal{I}(\psi)$, which implies $\Gamma \sqsubseteq_\infty \psi$.

First, we state our interpretation:

$$\begin{aligned} D^{\mathcal{I}} &= \text{consistent finite sets of subformulas of the TBox } \mathcal{T} \\ \mathcal{I}(\psi) &= \{\Gamma \mid \Gamma \sqsubseteq_\infty \psi\} \\ \mathcal{I}^1(r)(\Gamma, \Delta) &\leftrightarrow \Gamma \Vdash \exists r.\top \\ \mathcal{I}^2(r)(\Gamma, \Delta) &\leftrightarrow \Delta \text{ consists of all } \delta \text{ in the set} \\ &\qquad \text{of subformulas of } \mathcal{T} \text{ for which } \Gamma \Vdash \forall r.\delta \end{aligned}$$

The proof consists of two elements:

(a) Show that $\mathcal{I}$ satisfies the conditions for the fixpoint interpretation

(b) Show that $\mathcal{I}$ is indeed a greatest interpretation: $\mathcal{J}(\psi) \subseteq \mathcal{I}(\psi)$ for all $\mathcal{J}$ and all ϕ.

1. $\psi = \bot$:
   (a) We want to show that $\mathcal{I}(\bot) = \emptyset$. This is the case, since the domain only contains *consistent* sets of sentences.
   (b) We want to show that $\mathcal{J}(\bot) \subseteq \mathcal{I}(\bot)$. This follows from $\mathcal{J}(\bot) = \mathcal{I}(\bot) = \emptyset$.

2. $\psi = \neg P$:
   (a) We need to prove: $\mathcal{I}(\neg P) = D^{\mathcal{I}} - \mathcal{I}(P)$
   Take $\Gamma \in \mathcal{I}(\neg P)$, then by assumption on $\mathcal{J}$: $\mathcal{I}(P) = \mathcal{J}(P)$. Then, we get $\Gamma \in \mathcal{J}(\neg P) = D^{\mathcal{I}} - \mathcal{J}(P) = D^{\mathcal{I}} - \mathcal{I}(P)$.
   (b) Then we need to show that $\mathcal{J}(\neg P) \subseteq \mathcal{I}(\neg P)$.
   $\Gamma \in \mathcal{J}(\neg P)$ and thus $\Gamma \in D^{\mathcal{I}} - \mathcal{J}(P) = D^{\mathcal{I}} - \mathcal{I}(P) = \mathcal{I}(\neg P)$.

3. $\psi = \exists r.\top$:
   (a) We need to show that $\mathcal{I}(\exists r.\top) = \{x \mid \exists y \in D^{\mathcal{I}}. \mathcal{I}^1(r)(x, y)\}$
   Take $\Gamma \in \mathcal{I}(\exists r.\top)$, then $\Gamma \sqsubseteq_\infty \exists r.\top$. By generation, we get $\Gamma \Vdash \exists r.\top$. By the definition of our interpretation, we then have $\Delta \in D^{\mathcal{I}}$ such that $\mathcal{I}^1(r)(\Gamma, \Delta)$.
   For the other direction, assume Γ and Δ such that $\mathcal{I}^1(r)(\Gamma, \Delta)$. Then $\Gamma \Vdash \exists r.\top$ and $\Gamma \sqsubseteq_\infty \exists r.\top$, and thus $\Gamma \in \mathcal{I}(\exists r.\top)$.
   (b) Then show that $\mathcal{J}(\exists r.\top) \subseteq \mathcal{I}(\exists r.\top)$.
   Take $\Gamma \in \mathcal{J}(\exists r.\top)$ then there exists a Δ such that $\mathcal{I}^1(r)(\Gamma, \Delta)$. Then, by the definition of $\mathcal{I}^1(r)$, we have $\Gamma \sqsubseteq_\infty \exists r.\top$.

□

3.3 $\mathcal{ALE}'$

In this section we consider the logic $\mathcal{ALE}'$, obtained by adding the full existential constructor to $\mathcal{AL}'$. This logic is only one step removed from $\mathcal{ALC}$. The only difference between the two logics is the disjunction. However, this does simplify the logic significantly, and we can easily use the strategy we have been using so far.

We introduce the following grammar:

$$\phi ::= \top \mid \bot \mid X \mid P \mid \neg P \mid \phi \sqcap \phi \mid \exists r.\phi \mid \forall r.\phi$$

Then, we add the following rule to the sequent calculus described in the previous section:

$$\frac{\phi \sqsubseteq_n \chi}{\Gamma', \exists r.\phi \sqsubseteq_n \exists r.\chi}\ \exists$$

We extend the *greatest fixpoint interpretation* in such a way that any function $\mathcal{J}$ needs to fulfill the condition:

$$\mathcal{J}(\exists r.\phi) = \{x \mid \exists y \in \mathcal{J}(\phi). \mathcal{I}^1(r)(x,y)\}$$

The definition of $\Vdash$ stays the same as defined in 3.2.2, and therefore we do not need to prove the partial cut lemma 3.2.3 again. We can use these directly for the generation and soundness proof.

3.3.1 Soundness

The generation lemma is adjusted to include the case of $\exists r.\psi$.

Lemma 3.3.1 (Generation)**.** Suppose $n > 0$:

7. $\Gamma \sqsubseteq_n \exists r.\psi$ iff there is a subformula ρ of the TBox or of Γ,$\exists r.\psi$ such that $\Gamma \Vdash \exists r.\rho$ and $\rho \sqsubseteq_n \psi$.

Proof. All other cases remain the same, so we only prove point 7:

7. To show: $\Gamma \sqsubseteq_n \exists r.\psi$ iff $\Gamma \Vdash \exists r.\rho$ and $\rho \sqsubseteq_n \psi$.

   The right-to-left direction is the following derivation:

$$\frac{\Gamma \Vdash \exists r.\rho \qquad \dfrac{\rho \sqsubseteq_n \psi}{\exists r.\rho \sqsubseteq_n \exists r.\psi}\ \exists}{\Gamma \sqsubseteq_n \exists r.\psi}\ pc$$

   For the other direction, assume $\Gamma \sqsubseteq_n \exists r.\psi$. Again, we perform an induction on the length of this derivation. The last rule used is one of the following Ax, $Ax\bot$, $\sqcap L$, $DefL$, $\exists$. Our goal is to find the ρ such that $\Gamma \Vdash \exists r.\rho$ and $\rho \sqsubseteq_n \psi$.

- Ax:

$$\frac{}{\Gamma', \exists r.\psi \sqsubseteq_n \exists r.\psi}\ Ax$$

  then, $\rho = \psi$. Since $\Gamma', \exists r.\psi \Vdash \exists r.\psi$ and $\psi \sqsubseteq_n \psi$.

- $Ax\bot$:

$$\frac{}{\Gamma', \bot \sqsubseteq_n \exists r.\psi}\ Ax\bot$$

  Then anything follows from $\Gamma', \bot$ and thus $\Gamma', \bot \Vdash \exists r.\psi$ and $\psi \sqsubseteq_n \psi$. Thus, $\rho = \psi$.

- $\sqcap L$:

$$\frac{\Gamma', \phi_1, \phi_2 \sqsubseteq_n \exists r.\psi}{\Gamma', \phi_1 \sqcap \phi_2 \sqsubseteq_n \exists r.\psi}\ \sqcap L$$

  Then we can use the same ρ as in the IH.

- $DefL$:

$$\frac{\Gamma', \phi(X) \sqsubseteq_n \exists r.\psi}{\Gamma', X \sqsubseteq_n \exists r.\psi}\ DefL$$

  We can use the same ρ as in the IH.

- $\exists$:

$$\frac{\phi \sqsubseteq \psi}{\Gamma', \exists r.\phi \sqsubseteq \exists r.\psi}\ \exists$$

  Then, from the induction hypothesis we have $\phi \sqsubseteq_n \psi$, and $\Gamma', \exists r.\phi \Vdash \exists r.\phi$ can be derived by Ax.

□

The following lemma does not change, but we state it again since we use it for the soundness proof.

Lemma 3.3.2. If $\Gamma \Vdash \psi$ then $\mathcal{I}^{\sqcap}(\Gamma) \subseteq \mathcal{I}(\psi)$ for any interpretation $\mathcal{I}$.

Theorem 3.3.3 (Soundness)**.** If $\Gamma \sqsubseteq_\infty \psi$ then $\Gamma \vDash_{gfp} \psi$.

Proof. This is equivalent to proving that for each ψ and greatest fixpoint interpretation $\mathcal{I}$ one has

$$\bigcup_{\Gamma \sqsubseteq_\infty \psi} \mathcal{I}^{\sqcap}(\Gamma) \subseteq \mathcal{I}(\psi)$$

And we take $\mathcal{J}(\psi) := \bigcup_{\Gamma \sqsubseteq_\infty \psi} \mathcal{I}^{\sqcap}(\Gamma)$, and show that $\mathcal{J}$ satisfies the greatest fixpoint conditions. We only consider the new case $\psi = \exists r.\psi_1$

- We need to show that $\mathcal{J}(\exists r.\psi_1) = \{x \mid \exists y \in \mathcal{J}(\phi).\mathcal{I}^1(r)(x,y)\}$

  Take $x \in \mathcal{J}(\exists r.\psi_1)$, then there is Γ such that $x \in \mathcal{I}^{\sqcap}(\Gamma)$ and $\Gamma \sqsubseteq_n \exists r.\psi_1$. The generation lemma gives us a ρ such that $\Gamma \Vdash \exists r.\rho$ and $\rho \sqsubseteq_\infty \psi_1$. This gives us $x \in \mathcal{I}(\exists r.\rho)$ so there exists a y such that $\mathcal{I}^1(r)(x,y)$ and $y \in \mathcal{I}(\rho)$. Then, since $\rho \sqsubseteq_\infty \psi_1$, $y \in \mathcal{J}(\psi_1)$ by the definition of $\mathcal{J}$.

  Now assume x, y such that $\mathcal{I}^1(r)(x,y)$ and $y \in \mathcal{J}(\psi_1)$. Then for some Γ such that $\Gamma \sqsubseteq_\infty \psi_1$ we have $y \in \mathcal{I}^{\sqcap}(\Gamma) = \mathcal{I}(\sqcap\Gamma)$, where $\sqcap\Gamma$ is the conjunction of all formulas $\gamma \in \Gamma$. We derive the following:

$$\cfrac{\cfrac{\Gamma \sqsubseteq_\infty \psi_1}{\sqcap\Gamma \sqsubseteq_\infty \psi_1}\ \sqcap L}{\exists r.\sqcap\Gamma \sqsubseteq_\infty \exists r.\psi_1}\ \exists$$

  Then, since $y \in \mathcal{I}(\sqcap\Gamma)$, we have $x \in \mathcal{I}(\exists r.\sqcap\Gamma)$, and therefore $x \in \mathcal{J}(\exists r.\psi_1)$.

□

3.3.2 Completeness

Then, we move on to the completeness proof. The main idea of our model remains the same, but we adjust the interpretation $\mathcal{I}^1(r)(\Gamma,\Delta)$ to fit our new constructor $\exists r.\phi$.

Theorem 3.3.4 (Completeness)**.** If $\Gamma \vDash_{gfp} \psi$ then $\Gamma \sqsubseteq_\infty \psi$.

Proof. The strategy for the proof is the same as chapter 2.

$$\begin{aligned}
D^{\mathcal{I}} &= \text{consistent finite sets of subformulas of the TBox}\\
\mathcal{I}(\psi) &= \{\Gamma \mid \Gamma \sqsubseteq_\infty \psi\}\\
\mathcal{I}^1(r)(\Gamma,\Delta) &\leftrightarrow \Gamma \Vdash \exists r.\sqcap\Delta\\
\mathcal{I}^2(r)(\Gamma,\Delta) &\leftrightarrow \Delta \text{ consists of all } \delta \text{ in the set}\\
&\qquad \text{of subformulas of } \mathcal{T} \text{ for which } \Gamma \Vdash \forall r.\delta
\end{aligned}$$

We only consider the case of $\exists r.\psi_1$, since this is the only one that changed.

- $\psi = \exists r.\psi_1$:

  1. First, we check whether $\mathcal{I}(\exists r.\psi_1) = \{x \mid \exists y \in \mathcal{I}(\psi_1).\mathcal{I}^1(r)(x,y)\}$
     - $\Gamma \in \mathcal{I}(\exists r.\psi_1)$ and thus $\Gamma \sqsubseteq_\infty \exists r.\psi_1$. Then by the generation lemma, there is a formula ρ such that $\Gamma \Vdash \exists r.\rho$ and $\rho \sqsubseteq_\infty \phi$. This gives us $\{\rho\} \in \mathcal{I}(\phi)$ and $\mathcal{I}^1(r)(\Gamma,\{\rho\})$, which is what we want.
     - Assume $\Gamma, \Delta \in D^{\mathcal{I}}$ such that $\mathcal{I}^1(r)(\Gamma,\Delta)$ and $\Delta \in \mathcal{I}(\psi_1)$. Then $\Delta \sqsubseteq_\infty \psi_1$, and thus $\rho \sqsubseteq_\infty \psi$ for $\rho = \sqcap\Delta$. The definition of $\mathcal{I}^1(r)$ gives us $\Gamma \Vdash \exists r.\rho$. From the generation lemma, it follows $\Gamma \sqsubseteq_\infty \exists r.\psi_1$.
  2. Now, check $\Gamma \in \mathcal{J}(\exists r.\psi_1)$ implies $\Gamma \sqsubseteq_\infty \exists r.\psi_1$

This is done by induction on the complexity of the formula. Thus, we assume that this holds for ψ_1. Now assume $\Gamma \in \mathcal{J}(\exists r.\psi_1)$. By the definition of any interpretation $\mathcal{J}$, there is a Δ such that $\mathcal{I}^1(r)(\Gamma, \Delta)$ and $\Delta \in \mathcal{J}(\psi_1)$. By IH, we get $\Delta \sqsubseteq_\infty \psi_1$, and thus $\rho \sqsubseteq_\infty \psi$ for $\rho = \sqcap \Delta$. Since the $\mathcal{I}^1(r)(\Gamma, \Delta)$ gives us $\Gamma \Vdash \exists r.\rho$ and thus generation gives us $\Gamma \sqsubseteq_\infty \exists r.\psi_1$.

□

Chapter 4

Quantifier-free Description Logic

4.1 Introduction

In chapters 2 and 3, the logics used were intuitionistic in the sense that the quantifiers $\forall$ and $\exists$ were not interdefinable. Therefore, the sequents in our calculus were of the form $\Gamma \sqsubseteq_n \phi$, with a set of formulas on the left and one formula on the right. In this section, we show a strategy for sequents of the form $\Gamma \sqsubseteq_n \Delta$.

We study the following logic:

$$\phi ::= \top \mid \bot \mid P \mid X \mid \phi \sqcap \phi \mid \phi \sqcup \phi$$

This is not a commonly used DL. The absence of quantifiers causes for limits in its expressive power.

One can also wonder whether we actually want a logic of this form to have circular definition. A circular definition in this logic is of the form:

$$Parent = Parent \sqcap Person$$

However, why would we want to state that a Parent is a Parent in the definition again?

Nevertheless, we dive into this logic, and prove soundness and completeness with respect to the greatest fixpoint semantics. We will see that a calculus where we allow for sets of formulas on the right, requires a different strategy in both proofs.

The logic has the following interpretation.

Definition 4.1.1 (Interpretation)**.** An interpretation $\mathcal{I}$ of this logic is a function mapping formulas to subsets of a non-empty domain $D^{\mathcal{I}}$, according to the following rules:

1. $\mathcal{I}(\top) = D^{\mathcal{I}}$
2. $\mathcal{I}(\bot) = \emptyset$
3. $\mathcal{I}(X) = \mathcal{I}(\phi(X))$
4. $\mathcal{I}(\phi \sqcap \psi) = \mathcal{I}(\phi) \cap \mathcal{I}(\psi)$
5. $\mathcal{I}(\phi \sqcup \psi) = \mathcal{I}(\phi) \cup \mathcal{I}(\psi)$

Then we add two rules for the $\sqcup$-operator, and make minor adjustments to the already existing rules:

$$\frac{}{\Gamma, \phi \sqsubseteq_n \Delta, \phi}\ Ax \qquad \frac{}{\Gamma, \bot \sqsubseteq_n \Delta}\ Ax\bot \qquad \frac{}{\Gamma \sqsubseteq_n \Delta, \top}\ Ax\top \qquad \frac{}{\Gamma \sqsubseteq_0 \Delta}\ start$$

$$\frac{\Gamma, \phi, \psi \sqsubseteq_n \Delta}{\Gamma, \phi \sqcap \psi \sqsubseteq_n \Delta}\ \sqcap L \qquad \frac{\Gamma \sqsubseteq_n \Delta, \psi \quad \Gamma \sqsubseteq_n \Delta, \chi}{\Gamma \sqsubseteq_n \Delta, \psi \sqcap \chi}\ \sqcap R$$

$$\frac{\Gamma, \phi(X) \sqsubseteq_n \Delta}{\Gamma, X \sqsubseteq_n \Delta}\ DefL \qquad \frac{\Gamma \sqsubseteq_n \Delta, \phi(X)}{\Gamma \sqsubseteq_{n+1} \Delta, X}\ DefR$$

$$\frac{\Gamma \sqsubseteq_n \Delta, \phi, \psi}{\Gamma \sqsubseteq_n \Delta, \phi \sqcup \psi}\ \sqcup R \qquad \frac{\Gamma, \phi \sqsubseteq_n \Delta \quad \Gamma, \psi \sqsubseteq_n \Delta}{\Gamma, \phi \sqcup \psi \sqsubseteq_n \Delta}\ \sqcup L$$

Then we have weakening in both sides of the axioms and it is thus evident that the following lemma holds.

Lemma 4.1.2. If $\Gamma \sqsubseteq_n \Delta$ then $\Gamma, \Lambda \sqsubseteq_n \Delta, \Psi$ for all $n \in \mathbf{N}$.

Then, we interpret a set of formulas Γ on the left of $\sqsubseteq_n$ as $\mathcal{I}^{\sqcap}(\Gamma) = \bigcap\{\mathcal{I}(\gamma) \mid \gamma \in \Gamma\}$, and the sets of formulas on the right of $\sqsubseteq_n$ as $\mathcal{I}^{\sqcup}(\Delta) = \bigcup\{\mathcal{I}(\delta) \mid \delta \in \Delta\}$.

We extend the definition of the greatest fixpoint semantics:

Definition 4.1.3. An interpretation of a TBox under the *greatest fixpoint semantics* is an interpretation $\mathcal{I}$ which has the further property that whenever $\mathcal{J}$ is a function mapping formulas over the TBox to subsets of $D^{\mathcal{I}}$ in such a way that

1. $\mathcal{J}(\top) = D^{\mathcal{I}}$
2. $\mathcal{J}(\bot) = \emptyset$
3. $\mathcal{J}(P) = \mathcal{I}(P)$
4. $\mathcal{J}(X) \subseteq \mathcal{J}(\phi(X))$
5. $\mathcal{J}(\phi \sqcap \psi) = \mathcal{J}(\phi) \cap \mathcal{J}(\psi)$
6. $\mathcal{J}(\phi \sqcup \psi) = \mathcal{J}(\phi) \cup \mathcal{J}(\psi)$

then $\mathcal{J}(\phi) \subseteq \mathcal{I}(\phi)$ for all ϕ.

4.2 Soundness

In order to prove soundness and completeness, we extend our definition of $\Vdash$ and adjust our generation lemma.

Definition 4.2.1. We write $\Gamma \Vdash \Delta$ if the judgement $\Gamma \sqsubseteq_n \Delta$ can be derived by using the rules Ax, $Ax\top$, $Ax\bot$, $\sqcap L$, $\sqcup L$ and $DefL$ for every $n \in \mathbf{N}$.

Lemma 4.2.2 (Partial Cut)**.** If $\Gamma \Vdash \psi, \Delta$ and $\Lambda, \psi \sqsubseteq_n \Psi$ then $\Gamma, \Lambda \sqsubseteq_n \Delta, \Psi$.

This lemma is proven by induction on the length of the derivation $\Gamma \Vdash \psi, \Delta$. This proof is similar to the one in the previous chapters, since the last rule used is only one of the left rules, or an axiom. Therefore, the sets of formulas Δ on the right side of $\Vdash$ does not affect the proof.

We now have sets of formulas on both sides, and therefore we need to prove the following for soundness and completeness: $\Gamma \sqsubseteq_\infty \Delta$ iff $\Gamma \vDash_{gfp} \Delta$. This entails we need to adjust our generation lemma to include sets of sentences, as well.

Lemma 4.2.3 (Generation)**.** Suppose $n > 0$:

1. If Δ is a set containing only propositional letters P, $\top$ or $\bot$, we have: $\Gamma \sqsubseteq_n \Delta$ iff $\Gamma \Vdash \Delta$
2. $\Gamma \sqsubseteq_{n+1} X, \Delta$ iff $\Gamma \sqsubseteq_n \phi(X), \Delta$
3. $\Gamma \sqsubseteq_n \psi_1 \sqcap \psi_2, \Delta$ iff $\Gamma \sqsubseteq_n \psi_1, \Delta$ and $\Gamma \sqsubseteq_n \psi_2, \Delta$
4. $\Gamma \sqsubseteq_n \psi_1 \sqcup \psi_2, \Delta$ iff $\Gamma \sqsubseteq_n \psi_1, \psi_2, \Delta$

Proof. The strategy for this proof is basically the same as before. For the right-to-left direction, we apply the appropriate rule (2,3,4), or it is in the definition of $\Vdash$ (1). For the left-to-right direction, in case 2, 3 and 4 we again perform an induction on the length of the derivation. However, the last rule used can now be any of the rules.

1. These cases require the same reasoning as we have seen before: the last rule used can only possibly one of Ax, $Ax\top$, $Ax\bot$, $\sqcap L$, $DefL$, $\sqcup L$, and those are the ones defined in $\Vdash$.
2. $\Gamma \sqsubseteq_{n+1} X, \Delta$ iff $\Gamma \sqsubseteq_n \phi(X), \Delta$.

   The right-to-left direction is just applying the rule $DefR$. For the left-to-right direction, we prove by induction on the length of the derivation $\Gamma \sqsubseteq_{n+1} X, \Delta$. As a base case we assume that the last rule used was Ax, $Ax\top$, or $Ax\bot$:

   - Ax: There are two cases, either X is not the principal formula or it is. Assume it is then:

     $$\frac{}{\Gamma', X \sqsubseteq_{n+1} X, \Delta}\ Ax$$

     We derive the following:

     $$\frac{\dfrac{}{X, \Gamma' \Vdash X, \Delta}\ Ax \qquad \dfrac{\phi(X) \sqsubseteq_n \phi(X)}{X \sqsubseteq_n \phi(X)}\ DefL}{X, \Gamma' \sqsubseteq_n \phi(X), \Delta}\ pc$$

     Assume it is not, then:

     $$\frac{}{\Gamma', \psi \sqsubseteq_{n+1} X, \psi, \Delta}\ Ax$$

     Then our wanted result: $\Gamma', \psi \sqsubseteq_n \phi(X), \psi, \Delta$ is just an instance of Ax.

- $Ax\bot$:

$$\frac{}{\Gamma', \bot \sqsubseteq_{n+1} X, \Delta} \; Ax\bot$$

Then the judgement we want to derive $\Gamma', \bot \sqsubseteq_n \phi(X), \Delta$, is an instance of $Ax\bot$.

- $Ax\top$:

$$\frac{}{\Gamma \sqsubseteq_{n+1} X, \top, \Delta} \; Ax\top$$

then the judgement we want to derive: $\Gamma \sqsubseteq_n \phi(X), \top, \Delta$ is an instance of $Ax\top$.

For the inductive step, we can assume that X was either principal in the derivation, or it was not. If it was principal, then the last rule used is $DefR$ on the judgement $\Gamma \sqsubseteq_n \phi(X), \Delta$. This is what we want to derive, so we are done.

If X is not principal, the rule used before is either a one premise or a two-premise rule. We are going to assume that it was a rule applied on Γ, but it can be a rule applied on Δ, and it will be symmetric.

- Assume the last rule used was a one-premise rule L, and so the derivation is the following:

$$\frac{\Gamma' \sqsubseteq_{n+1} X, \Delta}{\Gamma \sqsubseteq_{n+1} X, \Delta} \; L$$

From the inductive step we can derive $\Gamma' \sqsubseteq_n \phi(X), \Delta$, and we apply the rule L on this to obtain $\Gamma \sqsubseteq_n \phi(X), \Delta$.

- Then we assume the last rule used was a two premise rule L, and so the derivation is the following:

$$\frac{\Gamma' \sqsubseteq_{n+1} X, \Delta \quad \Gamma'' \sqsubseteq_{n+1} X, \Delta}{\Gamma \sqsubseteq_{n+1} X, \Delta} \; L$$

Then from the inductive step we get: $\Gamma' \sqsubseteq_n \phi(X), \Delta$ and $\Gamma'' \sqsubseteq_n \phi(X), \Delta$. Applying L gives us $\Gamma \sqsubseteq_n \phi(X), \Delta$.

If the last rule used is one applied to Δ, this should be identical. There is only one case that might be of interest, and that is the case where the $DefR$ rule is last used, but X is not principal.

- We assume the following derivation:

$$\frac{\Gamma \sqsubseteq_{n+1} X, \Delta', \phi(Y)}{\Gamma \sqsubseteq_{n+2} X, \Delta', Y} \; DefR$$

The induction hypothesis gives us $\Gamma \sqsubseteq_n \phi(X), \Delta', \phi(Y)$

$$\frac{\Gamma \sqsubseteq_n \phi(X), \Delta', \phi(Y)}{\Gamma \sqsubseteq_{n+1} \phi(X), \Delta', Y} \; DefR$$

3. $\Gamma \sqsubseteq_n \psi_1 \sqcap \psi_2, \Delta$ iff $\Gamma \sqsubseteq_n \psi_1, \Delta$ and $\Gamma \sqsubseteq_n \psi_2, \Delta$

   For the right-to-left direction we can just apply $\sqcap R$, to obtain our wanted result.

   The left-to-right direction we prove by induction on the length of the derivation. First, assume the last rule used is one of the axioms:

   - Ax:

     In this case there are two options: either $\psi_1 \sqcap \psi_2$ is the axiomatic formula or it is not.

     First let us consider the case that it is:

     $$\frac{}{\Gamma', \psi_1 \sqcap \psi_2 \sqsubseteq_n \psi_1 \sqcap \psi_2, \Delta} \; Ax$$

     We can obtain the following derivation:

     $$\dfrac{\dfrac{}{\Gamma', \psi_1, \psi_2 \sqsubseteq_n \psi_1, \Delta} \; Ax}{\Gamma', \psi_1 \sqcap \psi_2 \sqsubseteq_n \psi_1, \Delta} \; \sqcap L \qquad \dfrac{\dfrac{}{\Gamma', \psi_1, \psi_2 \sqsubseteq_n \psi_2, \Delta} \; Ax}{\Gamma', \psi_1 \sqcap \psi_2 \sqsubseteq_n \psi_2, \Delta} \; \sqcap L$$

     Then for $\psi_1 \sqcap \psi_2$ not being principal:

     $$\frac{}{\Gamma', \phi \sqsubseteq_n \phi, \psi_1 \sqcap \psi_2, \Delta} \; Ax$$

     Our wanted results are instances of the axioms, and we are done.

   - For $Ax\bot$ and $Ax\top$ we obtain the same as in the previous derivation, that the wanted results can be derived because they are instances of the axioms.

   Then for the inductive case, we consider the case where $\phi_1 \sqcap \phi_2$ is principal, and where it is not. If it is not, the argument is identical to the one in the previous step.

   If $\phi_1 \sqcap \phi_2$ is principal, then the last rule used is $\sqcap R$, and then the step before applying this rule in the derivation gives our wanted result.

4. $\Gamma \sqsubseteq_n \psi_1 \sqcup \psi_2, \Delta$ iff $\Gamma \sqsubseteq_n \psi_1, \psi_2, \Delta$

   The right-to-left direction is just applying the rule $\sqcup R$.

   For the left-to-right direction we assume $\Gamma \sqsubseteq_n \psi_1 \sqcup \psi_2, \Delta$. For the base case we assume the last rule used is either Ax, $Ax\top$ or $Ax\bot$.

- Ax:

$$\frac{}{\Gamma', \psi_1 \sqcup \psi_2 \Vdash \psi_1 \sqcup \psi_2, \Delta}\ Ax$$

We derive our result in the following way:

$$\frac{\dfrac{}{\Gamma', \psi_1 \sqsubseteq_n \psi_1, \psi_2, \Delta}\ Ax \quad \dfrac{}{\Gamma', \psi_2 \sqsubseteq_n \psi_1, \psi_2, \Delta}\ Ax}{\Gamma', \psi_1 \sqcup \psi_2 \sqsubseteq_n \psi_1, \psi_2, \Delta}\ \sqcup L$$

For the case where $\psi_1 \sqcup \psi_2$ is not principal in the axiom, our wanted result is just another instance of the axiom: $\Gamma', \phi \sqsubseteq_n \phi, \psi_1, \psi_2, \Delta$.

- For $Ax\top$ our wanted result is an instance of the axiom $\Gamma \sqsubseteq_n \top, \psi_1, \psi_2, \Delta$.
- The same holds for $Ax\bot$: $\Gamma', \bot \sqsubseteq_n \psi_1, \psi_2, \Delta$.

Then for the inductive step, we have an identical case as above. The last rule used is $\sqcup L$, $\sqcap L$, $DefL$, $\sqcup R$, $\sqcap R$, or $DefR$. We can easily show using the induction hypopthesis that it holds.

□

Lemma 4.2.4. If $\Gamma \Vdash \Delta$ then $\mathcal{I}^{\sqcap}(\Gamma) \subseteq \mathcal{I}^{\sqcup}(\Delta)$ for any interpretation $\mathcal{I}$.

Proof. We have proven this lemma in chapter 2 by induction on the length of the derivation and considered the cases where the last rule used was Ax, $Ax\top$, $Ax\bot$, $\sqcap L$ and $DefL$. Thus, we only consider the rule $\sqcup L$:

$$\frac{\Gamma', \phi_1 \sqsubseteq_n \Delta \quad \Gamma', \phi_2 \sqsubseteq_n \Delta}{\Gamma', \phi_1 \sqcup \phi_2 \sqsubseteq_n \Delta}\ \sqcup L$$

By IH, we have that $\mathcal{I}^{\sqcap}(\Gamma') \cap \mathcal{I}(\phi_1) \subseteq \mathcal{I}^{\sqcup}(\Delta)$ and $\mathcal{I}^{\sqcap}(\Gamma') \cap \mathcal{I}(\phi_2) \subseteq \mathcal{I}^{\sqcup}(\Delta)$. By definition of intersection and union, we know:

$$\begin{aligned}
&(\mathcal{I}^{\sqcap}(\Gamma') \cap \mathcal{I}(\phi_1)) \cup (\mathcal{I}^{\sqcap}(\Gamma') \cap \mathcal{I}(\phi_2)) \\
=& \mathcal{I}^{\sqcap}(\Gamma') \cap (\mathcal{I}(\phi_1) \cup \mathcal{I}(\phi_2)) \\
=& \mathcal{I}^{\sqcap}(\Gamma') \cap \mathcal{I}(\phi_1 \sqcup \phi_2)
\end{aligned}$$

Therefore, we conclude: $\mathcal{I}^{\sqcap}(\Gamma') \cap \mathcal{I}(\phi_1 \sqcup \phi_2) \subseteq \mathcal{I}^{\sqcup}(\Delta)$. □

Now, we prove the soundness and completeness, but using a different strategy than the previous sections. Before, we wanted the following interpretation to satisfy the conditions for the greatest fixpoint interpretation:

$$\mathcal{J}(\psi) := \bigcup_{\Gamma \sqsubseteq_\infty \psi} \mathcal{I}^{\sqcap}(\Gamma)$$

This means that we need to prove that $\mathcal{J}(\phi_1 \sqcup \phi_2) = \mathcal{J}(\phi_1) \cup \mathcal{J}(\phi_2)$, however this is not the case. Consider for example $\Gamma = \{\phi_1 \sqcup \phi_1\}$. Then, $\phi_1 \sqcup \phi_2 \sqsubseteq_\infty \phi_1 \sqcup \phi_2$, and thus $\mathcal{I}(\phi_1 \sqcup \phi_2) \subseteq \mathcal{J}(\phi_1 \sqcup \phi_2)$. However, the following does not, in general, hold: $\phi_1 \sqcup \phi_2 \sqsubseteq_n \phi_1$ or $\phi_1 \sqcup \phi_2 \sqsubseteq_n \phi_2$. Therefore, $\mathcal{I}(\phi_1 \sqcup \phi_2) \not\subseteq \mathcal{J}(\phi_1)$ and $\mathcal{I}(\phi_1 \sqcup \phi_2) \not\subseteq \mathcal{I}(\phi_1)$. So, $\mathcal{J}(\phi_1 \sqcup \phi_2) \neq \mathcal{J}(\phi_1) \cup \mathcal{J}(\phi_2)$. Let us prove soundness in a different way.

First, we introduce the following definitions.

Definition 4.2.5 (Pre-interpretation)**.** A function $\mathcal{J}$ mapping formulas to elements of a domain $D^{\mathcal{J}}$ is called a *pre-interpretation*, if it satisfies the following conditions:

- $\mathcal{J}(\top) = D^{\mathcal{J}}$
- $\mathcal{J}(\phi \sqcup \psi) = \mathcal{J}(\phi) \cup \mathcal{J}(\psi)$
- $\mathcal{J}(\bot) = \emptyset$
- $\mathcal{J}(\phi \sqcap \psi) = \mathcal{J}(\phi) \cap \mathcal{J}(\psi)$
- $\mathcal{J}(X) \supseteq \mathcal{J}(\phi(X))$

Definition 4.2.6 (Hierarchy of Pre-interpretations)**.** Given a greatest fixpoint interpretation $\mathcal{I}$, we define a *hierarchy or pre-interpretations* in the following way:

- $\mathcal{I}_\alpha(P) = \mathcal{I}(P)$
- $\mathcal{I}_\alpha(\phi \sqcup \psi) = \mathcal{I}_\alpha(\phi) \cup \mathcal{I}_\alpha(\psi)$
- $\mathcal{I}_\alpha(\top) = \mathcal{I}(\top)$
- $\mathcal{I}_0(X) = D^{\mathcal{I}}$
- $\mathcal{I}_\alpha(\bot) = \mathcal{I}(\bot)$
- $\mathcal{I}_{\alpha+1}(X) = \mathcal{I}_\alpha(\phi(X))$
- $\mathcal{I}_\alpha(\phi \sqcap \psi) = \mathcal{I}_\alpha(\phi) \cap \mathcal{I}_\alpha(\psi)$
- $\mathcal{I}_\lambda(X) = \bigcap_{\alpha<\lambda} \mathcal{I}_\alpha(X)$

These definitions are based on literature on fixpoints, found in [11]. We dive into this intuition in chapter 5.

We use this a hierarchy to prove soundness. We start by fixing a greatest fixpoint interpretation $\mathcal{I}$, and prove $\Gamma \sqsubseteq_\infty \Delta$ implies $\mathcal{I}^\sqcap(\Gamma) \subseteq \mathcal{I}_\alpha^\sqcup(\Delta)$ for all α. Since for every interpretation $\mathcal{I} = \mathcal{I}_\alpha$ holds for some α (this result can be found in [11]), and $\mathcal{I}$ is arbitrary, we get $\Gamma \vDash_{gfp} \Delta$.

Theorem 4.2.7 (Soundness)**.** If $\Gamma \sqsubseteq_\infty \Delta$ then $\Gamma \vDash_{gfp} \Delta$.

Proof. Take a greatest fixpoint interpretation $\mathcal{I}$. We prove $\Gamma \sqsubseteq_\infty \Delta$ implies $\mathcal{I}^\sqcap(\Gamma) \subseteq \mathcal{I}_\alpha^\sqcup(\Delta)$ for all α, and argue by induction on α and a sub-induction on the complexity of Δ. We consider the following two cases:

1. Δ only contains variables, propositional concept names, $\top$, and $\bot$.
2. Δ contains boolean formulas, $\psi_1 \sqcap \psi_2$, $\psi_1 \sqcup \psi_2$.

1. Assume Δ is a set of variables, propositional letters, $\top$, and $\bot$.

   We consider the base case $\alpha = 0$. If there are no variables in Δ, then $\mathcal{I}_\alpha^\sqcup(\Delta) = \mathcal{I}^\sqcup(\Delta)$. In this situation, we have $\Gamma \Vdash \Delta$ and by lemma 4.2.4 we have $\mathcal{I}^\sqcap(\Gamma) \subseteq \mathcal{I}^\sqcup(\Delta)$. If there is at least one variable X in Δ, then $\mathcal{I}_\alpha^\sqcup(\Delta) = D^{\mathcal{I}}$, and clearly $\mathcal{I}^\sqcap(\Gamma) \subseteq \mathcal{I}_\alpha^\sqcup(\Delta)$.

   Now let us move on to the inductive step. We assume that $\mathcal{I}^\sqcap(\Gamma) \subseteq \mathcal{I}_\beta^\sqcup(\Delta)$ holds for all $\beta < \alpha$, and prove that this holds for α

- If Δ contains only propositional letters, $\top$ and $\bot$, then by definition 4.2.6 $\mathcal{I}^{\sqcup}_{\alpha}(\Delta) = \mathcal{I}^{\sqcup}(\Delta)$. The generation lemma gives us $\Gamma \Vdash \Delta$, and by lemma 4.2.4, we get $\mathcal{I}^{\sqcap}(\Gamma) \subseteq \mathcal{I}^{\sqcup}(\Delta)$.
- Then consider the case where Δ contains variables: $\Delta = \{X_1, \ldots X_k\} \cup \Delta'$ and $\alpha > 0$.

  We assume $\Gamma \sqsubseteq_\infty \Delta$. By the generation lemma we get
  $\Gamma \sqsubseteq_\infty \phi_1(X_1), \ldots, \phi_k(X_k), \Delta'$ where each ϕ_i is associated to X_i.
  Our IH is as follows:

  $$\mathcal{I}^{\sqcap}(\Gamma) \subseteq \mathcal{I}_\beta(\phi_1(X_1)) \cup \ldots \cup \mathcal{I}_\beta(\phi_k(X_k)) \cup \mathcal{I}^{\sqcup}(\Delta') \tag{4.1}$$

  for all $\beta < \alpha$.

  If α is a successor ordinal, so $\alpha = \beta + 1$, then we get our result by definition 4.2.6.

  If α is a limit ordinal, we get our result by (4.1) and:

  $$\bigcap_{\beta<\alpha} (\mathcal{I}_\beta(X_1) \cup \ldots \cup \mathcal{I}_\beta(X_k)) \subseteq \mathcal{I}_\alpha(X_1) \cup \ldots \cup \mathcal{I}_\alpha(X_k) \tag{4.2}$$

2. For this case, we assume for the induction hypothesis that $\mathcal{I}^{\sqcap}(\Gamma) \subseteq \mathcal{I}^{\sqcap}_{\alpha}(\Delta'')$ for any Δ'' with a complexity less than the complexity of Δ.

   - $\Delta = \{\psi_1 \sqcap \psi_2\} \cup \Delta'$

     By the generation lemma we get that $\Gamma \sqsubseteq_\infty \psi_1, \Delta'$ and $\Gamma \sqsubseteq_\infty \psi_2, \Delta'$, and by our induction hypothesis, we get that $\mathcal{I}^{\sqcap}(\Gamma) \subseteq \mathcal{I}_\alpha(\psi_1) \cup \mathcal{I}^{\sqcup}_{\alpha}(\Delta')$ and $\mathcal{I}^{\sqcap}(\Gamma) \subseteq \mathcal{I}_\alpha(\psi_1) \cup \mathcal{I}^{\sqcup}_{\alpha}(\Delta')$. Then, our definition 4.2.6 gives us:

     $$\begin{aligned} \mathcal{I}^{\sqcap}(\Gamma) &\subseteq (\mathcal{I}_\alpha(\psi_1) \cup \mathcal{I}^{\sqcup}_{\alpha}(\Delta')) \cap (\mathcal{I}_\alpha(\psi_1) \cup \mathcal{I}^{\sqcup}_{\alpha}(\Delta')) \\ &= \mathcal{I}^{\sqcup}_{\alpha}(\Delta') \cup (\mathcal{I}_\alpha(\psi_1) \cap \mathcal{I}_\alpha(\psi_2)) \\ &= \mathcal{I}^{\sqcup}_{\alpha}(\Delta') \cup \mathcal{I}_\alpha(\psi_1 \sqcap \psi_2) \end{aligned}$$

     This is what we wanted to prove.

   - $\Delta = \{\psi_1 \sqcup \psi_2\} \cup \Delta'$

     By the generation lemma we have $\Gamma \sqsubseteq_\infty \psi_1, \psi_2, \Delta'$. Our induction hypothesis, and our definition 4.2.6 we have that

     $$\begin{aligned} \mathcal{I}^{\sqcap}_{\alpha}(\Gamma) &\subseteq \mathcal{I}_\alpha(\psi_1) \cup \mathcal{I}_\alpha(\psi_2) \cup \mathcal{I}^{\sqcup}_{\alpha}(\Delta') \\ &= \mathcal{I}_\alpha(\psi_1 \sqcup \psi_2) \cup \mathcal{I}^{\sqcup}_{\alpha}(\Delta') \end{aligned}$$

□

4.3 Completeness

We require a new strategy for the proof of completeness. We first sketch why the previous strategy did not work. Consider the interpretation:

$$D^{\mathcal{I}} = \text{consistent sets of subformulas of the TBox}$$
$$\mathcal{I}(\psi) = \{\Gamma \mid \Gamma \sqsubseteq_\infty \psi\}$$

We need to prove that $\mathcal{I}(\phi_1 \sqcup \phi_2) = \mathcal{I}(\phi_1) \cup \mathcal{I}(\phi_2)$. This means that if $\Gamma \in \mathcal{I}(\phi_1 \sqcup \phi_2)$, then $\Gamma \in \mathcal{I}(\phi_1)$ or $\Gamma \in \mathcal{I}(\phi_2)$. In our interpretation, this translates to: if $\Gamma \sqsubseteq_\infty \phi_1 \sqcup \phi_2$, then $\Gamma \sqsubseteq_\infty \phi_1$ or $\Gamma \sqsubseteq_\infty \phi_2$. However, if we take $\Gamma = \{\phi_1 \sqcup \phi_2\}$, we see that this is not the case. Thus, let us move on to our adjusted proof.

Theorem 4.3.1 (Completeness)**.** If $\Gamma \vDash_{gfp} \Delta$ then $\Gamma \sqsubseteq_\infty \Delta$.

Proof. Assuming $\Gamma \vDash_{gfp} \Delta$, we prove by induction on n that $\Gamma \sqsubseteq_n \Delta$ for all $n \in \mathbf{N}$.

The base case $n = 0$, is trivial, since $\Gamma \sqsubseteq_0 \Delta$ holds for all Γ, Δ by the start-axiom. Assume $\Gamma \sqsubseteq_n \Delta$, and prove $\Gamma \sqsubseteq_{n+1}$ by a sub-induction on the sum of the complexities of the formulas in Δ.

1. Δ only contains variables and propositional letters, $\top$, and $\bot$.
2. Δ contains formulas of the form $\psi_1 \sqcap \psi_2$, $\psi_1 \sqcup \psi_2$.

1.
   - Assume Δ only contains propositional letters, $\top$ and $\bot$. Then by the generation lemma, we have $\Gamma \Vdash \Delta$. Since this derivation does not use the $DefR$ rule by definition of $\Vdash$, all leaves of this derivation are Ax rules, and none of them are *start*. This means that we have $\Gamma \sqsubseteq_n P$ for any n so also $n + 1$.
   - Now assume $\Delta = \{X_1, \ldots, X_k\} \cup \Delta'$ and assume $\Gamma \sqsubseteq_n \Delta$. By assumption, we have $\mathcal{I}^{\sqcap}(\Gamma) \subseteq \mathcal{I}^{\sqcup}(\Delta)$. By definition of $\mathcal{I}$ have $\mathcal{I}(X_i) \subseteq \mathcal{I}(\phi_i(X))$ for all variables $X_i \in \Delta$. Transitivity of $\subseteq$ gives us $\mathcal{I}^{\sqcap}(\Gamma) \subseteq \mathcal{I}(\phi_1(X_1)) \cup \ldots \cup \mathcal{I}(\phi_k(X_k)) \cup \mathcal{I}^{\sqcup}(\Delta')$. The, the IH gives us $\Gamma \sqsubseteq_n \phi_1(X_1), \ldots, \phi_k(X_k), \Delta'$. Applying $DefR$, k times, to obtain $\Gamma \sqsubseteq_{n+k} X_1, \ldots, X_k, \Delta'$, and thus $\Gamma \sqsubseteq_{n+1} X_1, \ldots, X_K, \Delta'$.

2. Assume Δ contains the formulas $\psi_1 \sqcap \psi_2$ or $\psi_1 \sqcup \psi_2$.
   - Assume $\Gamma \sqsubseteq_n \psi_1 \sqcap \psi_2, \Delta'$. By the generation lemma, we obtain $\Gamma \sqsubseteq_n \psi_1, \Delta'$ and $\Gamma \sqsubseteq_n \psi_2, \Delta'$. The induction hypothesis for Δ gives us $\Gamma \sqsubseteq_{n+1} \psi_1, \Delta'$ and $\Gamma \sqsubseteq_{n+1} \psi_2, \Delta'$, and applying $\sqcap R$ gives us $\Gamma \sqsubseteq_{n+1} \psi_1 \sqcap \psi_2, \Delta'$.
   - Assume $\Gamma \sqsubseteq_n \psi_1 \sqcup \psi_2, \Delta'$. By the generation lemma, we obtain $\Gamma \sqsubseteq_n \psi_1, \psi_2, \Delta'$. The induction hypothesis gives $\Gamma \sqsubseteq_{n+1} \psi_1, \psi_2, \Delta'$ and applying $\sqcup R$ gives us $\Gamma \sqsubseteq_{n+1} \psi_1 \sqcup \psi_2, \Delta'$.

□

Chapter 5

Explicit Fixpoints

5.1 Introduction

Up to now, we have dealt with different small description logics allowing cyclic TBoxes, and we used greatest fixpoint semantics to obtain unique interpretations. However, not every application of a TBox benefits from a greatest fixpoint interpretation. We sketch an argument given in [10].

Greatest fixpoint semantics assign the greatest possible solution to the equation $X = f(X)$, and are primarily meant to interpret non-well-founded definitions. The example used in [10] is the class of *streams*: we start by having a node, and keep adding a successor that is a stream, obtaining an infinite sequence of nodes. The definition of a stream is then given to be:

$$stream = node \sqcap \leq 1 successor.\top \sqcap \exists successor.stream$$

In this definition we only want infinite sequences to be streams, and therefore would assign a greatest fixpoint interpretation to the concept name stream.

Then, let us consider the class of *lists*. The definition of a list is similar to the definition of a stream apart from the property that an empty-list is also considered a list. Thus we get:

$$list = emptylist \sqcup (node \sqcap \leq 1 successor.\top \sqcap \exists successor.list)$$

Here, an interpretation for *list* can be a finite subset of the domain. This class benefits there from least fixpoint semantics, where equations $X = f(X)$ are assigned the smallest possible solution.

These are two ways to interpret circular definitions in a unique way. Nonetheless, there are cases where uniqueness is not fitting. Take the following two concepts descriptions:

$$human = mammal \sqcap \exists parent.\top \sqcap \forall parent.human$$
$$horse = mammal \sqcap \exists parent.\top \sqcap \forall parent.horse$$

Then, according to the greatest fixpoint semantics $horse^{\mathcal{I}} = human^{\mathcal{I}}$, and in the least fixpoint interpretation $horse^{\mathcal{I}} = human^{\mathcal{I}} = \emptyset^{\mathcal{I}}$. Thus, neither of these definitions are desirable. In this case, we prefer descriptive semantics: it is arbitrary how *human* and *horses* initially get assigned as long as the following holds:

$$\mathcal{I}(human) = \mathcal{I}(mammal \sqcap \exists parent.\top \sqcap \forall parent.human)$$
$$\mathcal{I}(horse) = \mathcal{I}(mammal \sqcap \exists parent.\top \sqcap \forall parent.horse)$$

In this case, humans and horses do not need to have the same interpretation.

In conclusion, all three semantics are useful for different types of definitions. Preferably, we get to use these in one interpretation. This brings us to the greatest fixpoint constructor ν, and the least fixpoint constructor μ.

The aim of this chapter is to sketch the intuition behind these different fixpoints, as well as to propose a calculus for a logic including all three types of semantics. All statements that are not proven but are marked as 'conjecture'.

5.2 Intuition

First, we obtain an intuition for the meaning of a greatest and least fixpoint operator, based on [11]. We start with the greatest fixpoint. Assume a domain $D^{\mathcal{I}}$ and a formula $\phi(X)$. Then, we search for a subset $A \subseteq D^{\mathcal{I}}$ where $\mathcal{I}(X) = A = \mathcal{I}(\phi(X \mapsto A))$, such that A is the greatest solution that satisfies this equation. We write $\mathcal{I}(\phi(X \mapsto B))$ to mean: the interpretation of $\phi(X)$ where the interpretation of X is mapped to $B \subseteq D^{\mathcal{I}}$.

This searching procedure can be described in the following way. Consider the biggest available collection: the domain $D^{\mathcal{I}}$. Then we plug that into the formula and find its interpretation $\mathcal{I}(\phi(X \mapsto D^{\mathcal{I}}))$ (from now on we write this as $\mathcal{I}(\phi(D^{\mathcal{I}}))$. Then there are two possible outcomes, either $\mathcal{I}(\phi(D^{\mathcal{I}})) = D^{\mathcal{I}}$ or $\mathcal{I}(\phi(D^{\mathcal{I}})) \subset D^{\mathcal{I}}$. If it is the former, then we are done, and our greatest fixpoint is $D^{\mathcal{I}}$. If it is the latter we continue the procedure, and consider $\mathcal{I}(\phi(\phi(D^{\mathcal{I}})))$, etc. The idea is that there is an ordinal α, finite or infinite, such that $\mathcal{I}(\phi^{\alpha}(D^{\mathcal{I}})) = \mathcal{I}(\phi^{\alpha+1}(D^{\mathcal{I}}))$. In summary:

$$D^{\mathcal{I}} \supseteq \mathcal{I}(\phi(D^{\mathcal{I}})) \supseteq \mathcal{I}(\phi(\phi(D^{\mathcal{I}}))) \supseteq \ldots \supseteq \mathcal{I}(\phi^{\alpha}(D^{\mathcal{I}})) = \mathcal{I}(\phi^{\alpha+1}(D^{\mathcal{I}})) = \ldots$$

Now, let us introduce some notation. We say $\nu^{\alpha}X.\phi(X)$, to mean the αth approximation of our fixpoint. This means that $\nu^{0}X.\phi(X) = D^{\mathcal{I}}$, $\nu^{1}X.\phi(X) = \mathcal{I}(\phi(X \mapsto \nu^{0}X.\phi(X))) = \mathcal{I}(\phi(D^{\mathcal{I}}))$, $\nu^{2}X.\phi(X) = \mathcal{I}(\phi(X \mapsto \nu^{1}X.\phi(X))) = \mathcal{I}(\phi(\phi(D^{\mathcal{I}})))$ etc. In the case sketched above, our greatest fixpoint is $\mathcal{I}(\phi^{\alpha}(D^{\mathcal{I}})))$, and this would be denoted as $\nu^{\alpha}X.\phi(X) = \mathcal{I}(\nu X.\phi(X))$.

This was a sketch for the definition of the approximation of the greatest fixpoint:

Definition 5.2.1.

- $\mathcal{I}^{0}(X) = D^{\mathcal{I}} = \nu^{0}X.\phi(X)$
- $\mathcal{I}^{\alpha+1}(X) = \mathcal{I}(\phi(X \mapsto \mathcal{I}^{\alpha}(X)))$ for ordinal n

- $\mathcal{I}^{\lambda}(X) = \bigcap_{\alpha<\lambda} \mathcal{I}^{n}(X)$ for limit ordinal λ and ordinal α

We want to relate this to the sequent calculus we have been discussing so far [12]. We consider the *start* rule and the rules for the greatest fixpoints:

$$\frac{}{\Gamma \sqsubseteq_0 \phi}\ start \qquad \frac{\Gamma, \phi(X) \sqsubseteq_n \psi}{\Gamma, X \sqsubseteq_n \psi}\ \nu DefL \qquad \frac{\Gamma \sqsubseteq_n \psi(X)}{\Gamma \sqsubseteq_{n+1} X}\ \nu DefR$$

Since we can always proof $\Lambda \sqsubseteq_0 \chi$ for all Λ, χ, we want a subset relation that always holds, such as $\mathcal{I}^{\sqcap}(\Gamma) \subseteq D^{\mathcal{I}}$. This is good since the domain is our 0th approximation: $\mathcal{I}(\phi^0(D^{\mathcal{I}})) = D^{\mathcal{I}}$. From here, we can applying our $\nu DefR$ rule to obtain the sequent $\Gamma \sqsubseteq_1 X$. This is also a judgement that is always derivable, and thus we want this to connect to the domain too: $\mathcal{I}^1(X) = D^{\mathcal{I}} = \mathcal{I}(\phi^0(D^{\mathcal{I}})) = \mathcal{I}^0(\phi(X))$. We put the notes 0 and 1 in the interpretation to denote we use the interpretation of X on 'level' 1 ($\sqsubseteq_1$) and the interpretation of $\phi(X)$ on 'level' 0 ($\sqsubseteq_0$). Now let us assume we apply certain rules in such a way that we derive $\Gamma \sqsubseteq_1 \phi(X)$. This allows us to apply $DefR$ again:

$$\cfrac{\Gamma \sqsubseteq_0 \phi(X)}{\Gamma \sqsubseteq_1 X}\ \nu DefR$$
$$\vdots$$
$$\cfrac{\Gamma \sqsubseteq_1 \phi(X)}{\Gamma \sqsubseteq_2 X}\ DefR$$

In order to get a sound derivation system we want $\mathcal{I}^1(\phi(X)) = \mathcal{I}^2(X)$.

We have approximated the 0th step of X, and we now apply ϕ again. We say the $\mathcal{I}^1(\phi(X))$ is equal to ϕ applied to the 0th approximation of the fixpoint X, and thus $\mathcal{I}^1(\phi(X)) = \mathcal{I}(\phi(X \mapsto \nu^0 X.\phi(X))) = \mathcal{I}(\phi(D^{\mathcal{I}}))$. Then, the latter is the next approximation of X, and thus $\mathcal{I}^2(X) = \nu^1 X.\phi(X) = \mathcal{I}(\phi(X \mapsto \nu^0 X.\phi(X))) = \mathcal{I}^1(\phi(X))$.

This gives us the following pattern:

Conjecture 5.2.2.
1. $\Gamma \sqsubseteq_{n+1} X$ iff $\mathcal{I}^{\sqcap}(\Gamma) \subseteq \nu^n X.\phi(X)$
2. $\Gamma \sqsubseteq_{n+1} \phi(X)$ iff $\mathcal{I}^{\sqcap}(\Gamma) \subseteq \mathcal{I}(\phi^{n+1}(D^{\mathcal{I}})) = \mathcal{I}(\phi(X \mapsto \nu^n X.\phi(X)))$
3. $\Gamma \sqsubseteq_{\infty} X$ iff $\mathcal{I}^{\sqcap}(\Gamma) \subseteq \mathcal{I}(\nu X.\phi(X))$

Note that this pattern considers the greatest fixpoint appearing on the right, since that is when we go a step forward in our derivation. On the left we interpret the fixpoint immediately as it's greatest fixpoint and we just get $\mathcal{I}(\nu X.\phi(X)) = \mathcal{I}(\phi(\nu X.\phi(X)))$.

We carry on to the approximation of the least fixpoint interpretation, μ. Here we start looking for a fixpoint starting with the empty set. Let us say we find a fixpoint at the *nth* approximation. We write $\mathcal{I}(\phi(\emptyset))$ for $\mathcal{I}(\phi(X \mapsto \emptyset))$. The searching would look the following:

$$\emptyset \subseteq \mathcal{I}(\phi(\emptyset)) \subseteq \mathcal{I}(\phi(\phi(\emptyset))) \subseteq \ldots \subseteq \mathcal{I}(\phi^n(\emptyset)) = \mathcal{I}(\phi^{n+1}(\emptyset)) = \ldots$$

Here, we say that $\mu^0 X.\phi(X) = \emptyset$, $\mu^1 X.\phi(X) = \mathcal{I}(\phi(\emptyset))$, $\mu^2 X.\phi(X) = \mathcal{I}(\phi(X \mapsto \mu^1 X.\phi(X))) = \mathcal{I}(\phi(\phi(\emptyset)))$.

Using this sketch, we obtain a definition for the approximation of the least fixpoint:

Definition 5.2.3.
- $\mathcal{I}^0(X) = \emptyset = \mu^0 X.\phi(X)$
- $\mathcal{I}^{\alpha+1}(X) = \mathcal{I}(\phi(X \mapsto \mathcal{I}^{\alpha}(X)))$ for ordinal α
- $\mathcal{I}^{\lambda}(X) = \bigcup_{\alpha<\lambda} \mathcal{I}^n(X)$ for limit ordinal λ and ordinal α

We now introduce the following rules for the least fixpoint semantics:

$$\frac{\Gamma, \phi(X) \sqsubseteq_n \psi}{\Gamma, X \sqsubseteq_{n+1} \psi}\ \mu DefL \qquad\qquad \frac{\Gamma \sqsubseteq_n \phi(X)}{\Gamma \sqsubseteq_n X}\ \mu DefR$$

We still consider of the previously defined start rule. Note, that the step n to $n+1$ is done when we find the formula $\phi(X)$ left of $\sqsubseteq_n$ instead of the right.

Assume the following derivation:

$$\cfrac{\cfrac{}{\phi(X) \sqsubseteq_0 \psi}\ start}{X \sqsubseteq_1 \psi}\ \mu DefL$$
$$\vdots$$
$$\cfrac{\phi(X) \sqsubseteq_1 \psi}{X \sqsubseteq_2 \psi}\ \mu DefL$$

Then, similar to the greatest fixpoint semantics, we can assign interpretations per level in the following way: $\mathcal{I}^0(\phi(X)) = \emptyset$ and $\mathcal{I}^1(X) = \mu^0 X.\phi(X) = \emptyset$. This works for the first step since $\emptyset \subseteq \mathcal{I}(\psi)$ is a relation that holds for any ψ.

For the step from level 1 to 2, we assign the formulas the following way:

$$\mathcal{I}^1(\phi(X)) = \mathcal{I}(\phi(\emptyset)) = \mu^1 X.\phi(X) = \mathcal{I}^2(X)$$

We can generalize this to the following statement:

Conjecture 5.2.4.
1. $\phi(X) \sqsubseteq_{n+1} \psi$ iff $\mathcal{I}^{n+1}(\phi(X)) \subseteq \mathcal{I}(\psi)$
2. $X \sqsubseteq_{n+1} \psi$ iff $\mathcal{I}^{n+1}(X) \subseteq \mathcal{I}(\psi)$
3. $X \sqsubseteq_{\infty} \psi$ iff $\mathcal{I}(\mu X.\phi(X)) \subseteq \mathcal{I}(\psi)$

In this case the approximation of the least fixpoint is used for the left side of $\sqsubseteq$, but on the right side we just take the final interpretation of the least fixpoint immediately such that $\mathcal{I}(\mu X.\phi(X)) = \mathcal{I}(\phi(\mu X.\phi(X)))$.

Let us now introduce a logic and a sequent calculus accounting for both fixpoints as well as the descriptive semantics.

5.3 The Logic

We just consider the logic used in [12], and add the fixpoint constructs.

$$\phi ::= P \mid X \mid \phi \sqcap \phi \mid \exists r.\phi \mid \mu X.\phi(X) \mid \nu X.\phi(X)$$

where TBoxes are still of the form $X = \phi(X)$. The interpretation is defined the following way:

Definition 5.3.1 (Interpretation)**.** An interpretation $\mathcal{I}$ for this logic is a function mapping formulas to subsets of a non-empty domain $D^{\mathcal{I}}$, according to the following rules:

1. $\mathcal{I}(\phi \sqcap \psi) = \mathcal{I}(\phi) \cap \mathcal{I}(\psi)$
2. $\mathcal{I}(\exists r.\phi) = \{x \mid \exists y.\mathcal{I}(r)(x,y) \text{ and } y \in \mathcal{I}(\phi)\}$
3. $\mathcal{I}(X) = \mathcal{I}(\phi(X))$
4. $\mathcal{I}(\mu X.\phi(X)) = \bigcup_{\alpha<\lambda} \mathcal{I}(\phi(X \mapsto \mu^n X.\phi(X))) = \mathcal{I}(\phi(\mu X.\phi(X)))$ for limit ordinal λ and ordinal α.
5. $\mathcal{I}(\nu X.\phi(X)) = \bigcap_{\alpha<\lambda} \mathcal{I}(\phi(X \mapsto \nu^\alpha X.\phi(X))) = \mathcal{I}(\phi(\nu X.\phi(X)))$ for limit ordinal λ and ordinal α.

The interpretation of 1-3 is just as we know so far. For points 4 and 5, we use the interpretation as given by definitions 5.2.3 and 5.2.1, respectively. In this interpretation, we expect this λ to be finite, due to the correspondence with our family of relations $\sqsubseteq_n$, but will not be further proven in this thesis.

We have the following derivation system:

$$\frac{}{\Gamma, \phi \sqsubseteq_n \phi}\ Ax \qquad \frac{}{\Gamma \sqsubseteq_0 \phi}\ start \qquad \frac{\Gamma, \phi, \psi \sqsubseteq_n \chi}{\Gamma, \phi \sqcap \psi \sqsubseteq_n \chi}\ \sqcap L \qquad \frac{\Gamma \sqsubseteq_n \psi \quad \Gamma \sqsubseteq_n \chi}{\Gamma \sqsubseteq_n \psi \sqcap \chi}\ \sqcap R$$

$$\frac{\phi \sqsubseteq_n \psi}{\Gamma, \exists r.\phi \sqsubseteq_n \exists r.\psi}\ \exists \qquad \frac{\Gamma, \phi(\mu X.\phi(X)) \sqsubseteq_n \psi}{\Gamma, \mu X.\phi(X) \sqsubseteq_{n+1} \psi}\ \mu DefL \qquad \frac{\Gamma \sqsubseteq_n \psi(\mu X.\psi(X))}{\Gamma \sqsubseteq_n \mu X.\psi(X)}\ \mu DefR$$

$$\frac{\Gamma, \phi(\nu X.\phi(X)) \sqsubseteq_n \psi}{\Gamma, \nu X.\phi(X) \sqsubseteq_n \psi}\ \nu DefL \qquad \frac{\Gamma \sqsubseteq_n \psi(\nu X.\psi(X))}{\Gamma \sqsubseteq_{n+1} \nu X.\psi(X)}\ \nu DefR$$

$$\frac{\Gamma, \phi(X) \sqsubseteq_n \psi}{\Gamma, X \sqsubseteq_n \psi}\ DefL \qquad \frac{\Gamma \sqsubseteq_n \psi(X)}{\Gamma \sqsubseteq_n X}\ DefR$$

Using an example we show that our system works the way we want it to work. Let $X = \mu X.(P \sqcap \exists r.X)$ and $Y = \nu Y.(P \sqcap \exists r.Y)$. We can show $X \sqsubseteq_\infty Y$, but we do not have $Y \sqsubseteq_n X$ for any $n > 0$.

Let us show the first case, and consider it for $n = 4$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \sqcap \exists r.X \sqsubseteq_0 P \sqcap \exists r.Y}\ start}{X \sqsubseteq_1 P \sqcap \exists r.Y}\ \mu DefL}{X \sqsubseteq_2 Y}\ \nu DefR}{P, \exists r.X \sqsubseteq_2 \exists r.Y}\ \exists \qquad \dfrac{}{P, \exists r.X \sqsubseteq_2 P}\ Ax}{P, \exists r.X \sqsubseteq_2 P \sqcap \exists r.Y}\ \sqcap R}{P \sqcap \exists r.X \sqsubseteq_2 P \sqcap \exists r.Y}\ \sqcap L}{X \sqsubseteq_3 P \sqcap \exists r.Y}\ \mu DefL}{X \sqsubseteq_4 Y}\ \nu DefR$$

while for the second case, there is no way to get out of $\sqsubseteq_0$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P, \exists r.Y \sqsubseteq_0 P}\ start \qquad \dfrac{\dfrac{}{Y \sqsubseteq_0 X}\ start}{P, \exists r.Y \sqsubseteq_0 \exists r.X}\ \exists}{P, \exists r.Y \sqsubseteq_0 P \sqcap \exists r.X}\ \sqcap R}{P \sqcap \exists r.Y \sqsubseteq_0 P \sqcap \exists r.X}\ \sqcap L}{P \sqcap \exists r.Y \sqsubseteq_0 X}\ \mu DefR}{Y \sqsubseteq_0 X}\ \nu DefL$$

However, it is not as easy to prove soundness and completeness for this system. We can start off in a similar way, by giving a definition of $\Vdash$. We use this definition to prove that the rule

$$\dfrac{\Gamma \Vdash \psi \quad \psi \sqsubseteq_n \chi}{\Gamma \sqsubseteq_n \chi}\ pc$$

is admissible. Additionally, we use this definition to prove that $\Gamma \Vdash \psi$ implies $\mathcal{I}^{\sqcap}(\Gamma) \subseteq \mathcal{I}(\psi)$. Therefore, we do not want any of the rules in the definition of $\Vdash$ to contain the step from n to $n+1$. Furthermore, we can also not include the rule $\mu DefR$ to $\Vdash$, since the partial cut rule is not admissible anymore. Let us sketch that argument. Assume we have a cut formula $\mu X.\phi(X)$, that is principal in both the derivations $\Gamma \Vdash \psi(\mu X.\psi(X))$ and $\mu X.\psi(X)) \sqsubseteq_n \chi$:

$$\dfrac{\dfrac{\Gamma \Vdash \psi(\mu X.\psi(X))}{\Gamma \Vdash \mu X.\psi(X)}\ \mu DefR \qquad \dfrac{\psi(\mu X.\psi(X)) \sqsubseteq_n \chi}{\mu X.\psi(X) \sqsubseteq_{n+1} \chi}\ \mu DefL}{\Gamma \sqsubseteq_{n+1} \chi}\ pc$$

Then we transform this derivation to the following:

$$\dfrac{\Gamma \Vdash \psi(\mu X.\psi(X)) \quad \psi(\mu X.\psi(X)) \sqsubseteq_n \chi}{\Gamma \sqsubseteq_n \chi}\ pc$$

We get $\Gamma \sqsubseteq_n \chi$, but we needed to prove that $\Gamma \sqsubseteq_{n+1} \chi$.

Therefore we only add $\nu DefL$ to our definition of $\Vdash$.

Definition 5.3.2. We write $\Gamma \Vdash \psi$ if the judgement $\Gamma \sqsubseteq_n \psi$ is derived using the rules Ax, $\sqcap L$, $\nu DefL$ and $DefL$ for all $n \in \mathbf{N}$.

We can easily prove our lemma with that.

Lemma 5.3.3. If $\Gamma \Vdash \psi$ and $\psi \sqsubseteq_n \chi$ then $\Gamma \sqsubseteq_n \chi$.

Proof. We prove this by induction on the length of the derivation $\Gamma \Vdash \psi$ and consider the case where the last rule used is $\nu DefL$.

- $\nu DefL$:

$$\dfrac{\dfrac{\Gamma', \phi(\nu X.\phi(X)) \Vdash \psi}{\Gamma', \nu X.\phi(X) \Vdash \psi}\ \nu DefL \qquad \psi \sqsubseteq_n \chi}{\Gamma', \nu X.\phi(X) \sqsubseteq_n \chi}\ pc$$

  This derivation can be transformed to:

$$\dfrac{\dfrac{\Gamma', \phi(\nu X.\phi(X)) \Vdash \psi \quad \psi \sqsubseteq_n \chi}{\Gamma', \phi(\nu X.\phi(X)) \sqsubseteq_n \chi}\ pc}{\Gamma', \nu X.\phi(X) \sqsubseteq_n \chi}\ \nu DefL$$

□

Lemma 5.3.4. If $\Gamma \Vdash \psi$ then $\mathcal{I}^{\sqcap}(\Gamma) \subseteq \mathcal{I}(\psi)$.

Proof. Again, we do an induction on the length of the derivation, and only consider the case where the last rule used is $\nu DefL$.

- $\nu DefL$:

$$\dfrac{\Gamma', \phi(\nu X.\phi(X)) \sqsubseteq_n \psi}{\Gamma', \nu X.\phi(X) \sqsubseteq_n \psi}\ \nu DefL$$

  By induction hypothesis we obtain $\mathcal{I}^{\sqcap}(\Gamma') \cap \mathcal{I}(\phi(\nu X.\phi(X))) \subseteq \mathcal{I}(\psi)$. Then, because we are looking at the greatest fixpoint operator on the left of the derivation we have $\mathcal{I}(\phi(\nu X.\phi(X))) = \mathcal{I}(\nu X.\phi(X))$.

□

Having adjusted our definition of $\Vdash$, these lemmas still hold, but we are not entirely happy. One of the benefits of $\Vdash$ was that in the proof of lemma 5.3 we had considered all of our left rules, and would only focus on the right rules in our soundness definition using our generation lemma.

We can still get the following generation lemma, by a simple induction on the length of the derivation. We state this lemma as a conjecture, since we do not prove this.

Conjecture 5.3.5 (Generation)**.** Suppose $n > 0$:

1. $\Gamma \sqsubseteq_n \mu X.\psi(X)$ iff $\Gamma \sqsubseteq_n \psi(\mu X.\psi(X))$

2. $\Gamma \sqsubseteq_{n+1} \nu X.\psi(X)$ iff $\Gamma \sqsubseteq_n \psi(\nu X.\psi(X))$

3. $\Gamma \sqsubseteq_n X$ iff $\Gamma \sqsubseteq_n \psi(X)$

However, this is not enough to prove soundness and completeness. We state them in the following conjecture.

Conjecture 5.3.6 (Soundness & Completeness)**.** $\Gamma \sqsubseteq_\infty \psi$ iff $\Gamma \vDash \psi$.

We can not continue to prove this with the elements we do have, since we do not have soundness for the rule $\sqsubseteq_0$. Instead, we could resort strategies often used in modal mu-calculus, such as constructing an ill-founded proof and prune it to obtain a counter example. Or make a comparison to the cut-free sequent calculus for modal mu-calculus as presented in [2]. This is outside the scope of this thesis, and is left for future research.

Chapter 6

Discussion

The aim of this chapter is to discuss and summarize the found results. First, we discuss the difficulties that were found for applying the method of Hofmann to the more complex DLs including disjunction and full negation. Then, we summarize the work presented in this thesis and explain the personal contribution of the author. Finally, some recommendations are presented for future research.

6.1 Problems in $\mathcal{ALC}$

As described in the introduction, $\mathcal{ALC}$ is the DL containing all the logical connectives:

$$\phi ::= \top \mid \bot \mid P \mid X \mid \neg\phi \mid \phi \sqcap \phi \mid \phi \sqcup \phi \mid \exists r.\phi \mid \forall r.\phi$$

We will describe why the strategy as described in [12] does not work as easily for the full logic $\mathcal{ALC}$. First, we evaluate the possibility of adding the disjunction $\sqcup$ to the logic $\mathcal{ALE}$. Then, we add $\neg$ to the logic $\mathcal{AL}$.

6.1.1 Disjunction

Consider the following logic:

$$\phi ::= \top \mid \bot \mid P \mid \neg P \mid X \mid \phi \sqcap \phi \mid \phi \sqcup \phi \mid \exists r.\phi \mid \forall r.\phi$$

with the following interpretation:

Definition 6.1.1 (Interpretation)**.** An interpretation $\mathcal{I}$ of the logic $\mathcal{ALC}$ is a function mapping formulas to subsets of a non-empty domain $D^{\mathcal{I}}$, according to the following rules:

1. $\mathcal{I}(\top) = D^{\mathcal{I}}$
2. $\mathcal{I}(X) = \mathcal{I}(\phi(X))$
3. $\mathcal{I}(\phi \sqcap \psi) = \mathcal{I}(\phi) \cap \mathcal{I}(\psi)$
4. $\mathcal{I}(\phi \sqcup \psi) = \mathcal{I}(\phi) \cup \mathcal{I}(\psi)$

5. $\mathcal{I}(\exists r.\phi) = \{x \mid \exists y \in D^{\mathcal{I}}.\mathcal{I}(r)(x,y) \& y \in \mathcal{I}(\phi)\}$

6. $\mathcal{I}(\forall r.\phi) = \{x \mid \forall y \in D^{\mathcal{I}}.\mathcal{I}(r)(x,y) \rightarrow y \in \mathcal{I}(\phi)\}$

For this logic, it is desirable to have sets of sentences on both sides, in order to prove the following two equivalences:

$$\exists r.(\phi \sqcup \psi) \equiv \exists r.\phi \sqcup \exists r.\psi$$
$$\forall r.(\phi \sqcap \psi) \equiv \forall r.\phi \sqcup \forall r.\psi$$

This motivates the following sequent calculus:

$$\frac{}{\Gamma, \phi \sqsubseteq_n \Delta, \phi}\ Ax \qquad \frac{}{\Gamma, \bot \sqsubseteq_n \Delta}\ Ax\bot \qquad \frac{}{\Gamma \sqsubseteq_n \Delta, \top}\ Ax\top \qquad \frac{}{\Gamma \sqsubseteq_0 \Delta}\ start$$

$$\frac{\Gamma, \phi, \psi \sqsubseteq_n \Delta}{\Gamma, \phi \sqcap \psi \sqsubseteq_n \Delta}\ \sqcap L \qquad \frac{\Gamma \sqsubseteq_n \Delta, \psi \quad \Gamma \sqsubseteq_n \Delta, \chi}{\Gamma \sqsubseteq_n \Delta, \psi \sqcap \chi}\ \sqcap R \qquad \frac{\Gamma \sqsubseteq_n \rho}{\Psi, \forall r.\Gamma \sqsubseteq_n \forall r.\rho, \Delta}\ \forall$$

$$\frac{\Gamma, \phi(X) \sqsubseteq_n \Delta}{\Gamma, X \sqsubseteq_n \Delta}\ DefL \qquad \frac{\Gamma \sqsubseteq_n \Delta, \phi(X)}{\Gamma \sqsubseteq_{n+1} \Delta, X}\ DefR$$

$$\frac{\phi \sqsubseteq_n \Delta}{\Gamma, \exists r.\phi \sqsubseteq_n \exists r.\Delta, \Psi}\ \exists \qquad \frac{\Gamma \sqsubseteq_n \Delta, \phi, \psi}{\Gamma \sqsubseteq_n \Delta, \phi \sqcup \psi}\ \sqcup R \qquad \frac{\Gamma, \phi \sqsubseteq_n \Delta \quad \Gamma, \psi \sqsubseteq_n \Delta}{\Gamma, \phi \sqcup \psi \sqsubseteq_n \Delta}\ \sqcup L$$

In order to prove soundness and completeness, our generation lemma for $\exists$ and $\forall$ needs to contain sets of sentences on the right, that is, $\Gamma \sqsubseteq_n \exists r.\psi, \Delta$ and $\Gamma \sqsubseteq_n \forall r.\psi, \Delta$. However, this inversion is no longer as clear.

Let us consider the case of $\exists r.\phi$. In the previous sections, our generation lemma for $\exists$ was the following:

$$\Gamma \sqsubseteq_n \exists r.\phi \leftrightarrow \Gamma \Vdash \exists r.\rho \text{ and } \rho \sqsubseteq_n \phi \text{ for some } \rho$$

The direct translation from this, to involve sets, would be the following:

$$\Gamma \sqsubseteq_n \exists r.\phi, \Delta \leftrightarrow \Gamma \Vdash \exists r.\rho, \Delta \text{ and } \rho \sqsubseteq_n \phi \text{ for some } \rho$$

or, making the statement more similar to the rule $\exists$:

$$\Gamma \sqsubseteq_n \exists r.\Delta, \Psi \leftrightarrow \Gamma \Vdash \exists r.\Delta', \Psi \text{ and } \Delta' \sqsubseteq_n \Delta$$

However, it is not clear that this holds. Take for example the last rule used to be $\forall$. Then $\exists r.\rho$ has been derived using the weakening in the $\forall$ rule:

$$\frac{\Gamma' \sqsubseteq_n \psi}{\Lambda, \forall r.\Gamma' \sqsubseteq_n \forall r.\psi, \exists r.\phi, \Delta'} \ \forall$$

Of course, we can add any formula ρ during weakening. However, it is not given that there is a derivation $\Gamma \Vdash \exists r.\rho, \Delta'$. Let us say that we add all the rules except $DefR$ to the definition of $\Vdash$ so that we have the following definition:

Definition 6.1.2. We write $\Gamma \Vdash \Delta$ if the judgement $\Gamma \sqsubseteq_n \Delta$ can be derived by using the rules Ax, $Ax\top$, $Ax\bot$, $\sqcap L$, $\sqcup L$, $\sqcap R$, $\sqcup R$, $\exists$, $\forall$ and $DefL$ for every $n \in \mathbf{N}$.

We want to show that $\Gamma \Vdash \exists r.\rho, \Delta'$ holds, but there is nothing preventing Δ' to have been derived using the $DefR$ rule. A similar argument holds for the universal quantifier.

Of course, the reason causing problems is the fact that we want to prove soundness and completeness with respect to the greatest fixpoint semantics. Without the rules $DefR$ and $DefL$, we could have done a simple induction on the last rule used to prove that $\Gamma \sqsubseteq_n \exists r.\phi, \Delta$ implies $\mathcal{I}^{\sqcap}(\Gamma) \subseteq \mathcal{I}(\exists r.\phi) \cup \mathcal{I}^{\sqcup}(\Delta)$.

6.1.2 Negation

Adding the operator $\neg$ raises a similar issue but adds one more difficulty. Take the following language.

$$\phi ::= \top \mid \bot \mid P \mid \neg\phi \mid \phi \sqcap \phi \mid \forall r.\phi$$

Then we could define $\exists$ by: $\exists r.\phi \equiv \neg\forall r.\neg\phi$ and $\sqcup$ by: $\phi \sqcup \psi \equiv \neg(\neg\phi \sqcap \neg\phi)$.

We add the following rules for negation.

$$\frac{\Gamma \sqsubseteq_n \phi, \Delta}{\Gamma, \neg\phi \sqsubseteq_n \Delta} \ NegL \qquad\qquad \frac{\Gamma, \phi \sqsubseteq_n \Delta}{\Gamma \sqsubseteq_n \neg\phi, \Delta} \ NegR$$

With the appropriate interpretation:

$$\mathcal{I}(\neg\phi) = D^{\mathcal{I}} - \mathcal{I}(\phi)$$

This gives us another problem with the generation lemma. For example, consider the case $\Gamma \sqsubseteq_n \Delta$ where Δ consists only of propositional letters P, $\bot$ and $\top$. Our previous generation lemma, gives us $\Gamma \Vdash \Delta$. We prove this by induction on the derivation

$\Gamma \sqsubseteq_n \Delta$. Now consider the following derivation:

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \Gamma' \sqsubseteq_n \phi(X), \Delta\end{matrix}}{\Gamma' \sqsubseteq_{n+1} X, \Delta}\ DefR}{\Gamma', \neg X \sqsubseteq_{n+1} \Delta}\ NegL$$

Thus, $\Gamma \sqsubseteq_n \Delta$ has been derived using an application of the $DefR$ rule. It is possible to add more rules to the definition of $\Vdash$, but we do not want $\Vdash$ to contain the $DefR$ rule.

The strategy we have for proving soundness and completeness does not work, and we would have needed to make more adjustments.

6.2 Summary

In this thesis, we extended the framework as introduced in [12] to a sequent calculus for the frame-based description language $\mathcal{FL}_0$, and the attribute languages $\mathcal{AL}'$ and $\mathcal{ALE}'$, and we proved soundness and completeness with respect to the greatest fixpoint semantics. Where Hofmann provided us the tools for the connectives $\sqcap$ and $\exists r$, and we added connectives $\top$, $\bot$, $\forall$, and the atomic negation. From this, we conclude that this framework works for intuitionistic logics where the connectives are not interdefinable.

Then we applied this method to a quantifier-free DL containing the disjunction $\sqcup$. We have a sequent calculus for this logic that is sound and complete with respect to the greatest fixpoint semantics, but its proof required a new method using the definition of pre-interpretations.

Furthermore, we give the motivation for the need for a calculus that includes a least, greatest, and descriptive fixpoint all in one system. We propose a calculus and give an intuition for the relation between the interpretation and the sequent calculus. The actual proof of soundness and completeness is beyond the scope of this thesis.

6.3 Future research

In this work, the focus is on the standard connectives of DL and the cyclic extension of the TBox. Another interesting aspect is to extend the framework of the small DLs to include inverse roles or nominals (or other extensions) and to evaluate the consequences for the sequent calculus, as well as the soundness and completeness proofs. Other than extensions, one could include the ABox in the sequent calculi. Now, our calculi only reason about a given TBox and an ABox is not included.

Since description logic is so closely related to modal logic, there is value in comparing Hofmann's framework including least and greatest fixpoints to existing calculi for modal mu-calculus such as the methods described in [1] and [3].

In [12], Hofmann analyzes the decidability of the $\phi \vDash_{des} \psi$ relation for the logic $\mathcal{EL}$ he presented based on his presented calculus. In future research, one could analyze the complexity of the relations $\Gamma \vDash_{des} \psi$ for the logics presented in this thesis.

Moreover, in this thesis the TBox only consists of equations of the form $X = \phi(X)$. However, in applications of DL, the TBox often consists of subsumption relations $\phi \sqsubseteq \psi$. Hofmann considers a case where he includes these in his calculus as the so-called 'general inclusion axioms'. The addition of this notion can be studied for the logics presented here.

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